
DSC 140B - Quiz 03

Name: PID: **About the quizzes:**

- Quizzes in DSC 140B are *optional* and graded pass/fail.
- A score of 70% or higher earns a “pass” and 1.5 credits toward your final grade.
- If you don’t pass, no credits are earned, but it doesn’t hurt your grade.
- You have 30 minutes to complete the quiz.
- At least one of the questions below will be on an exam (probably with slight changes, such as different numbers).
- Unfortunately, we can’t answer clarifying questions during the quiz. If you think a question has a bug or is unclear, please let us know in a private post on Campuswire after the quiz, and we’ll take it into account when grading.

Problem 1.

Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ and let $\vec{v} = (2, -1, -1)^T$.

Is \vec{v} an eigenvector of A ?

- Yes, with eigenvalue $\lambda = 2$
- Yes, with eigenvalue $\lambda = 3$
- Yes, with eigenvalue $\lambda = 4$
- Yes, with eigenvalue $\lambda = 5$
- No, \vec{v} is not an eigenvector of A

Solution: Yes, with eigenvalue $\lambda = 2$.

We compute $A\vec{v}$:

$$A\vec{v} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 - 1 - 1 \\ 2 - 3 - 1 \\ 2 - 1 - 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 2\vec{v}$$

Since $A\vec{v} = 2\vec{v}$, the vector \vec{v} is an eigenvector with eigenvalue 2.

This was similar to Practice Problem 30.

Problem 2.

Let A be a matrix with eigenvector \vec{v} and corresponding eigenvalue -2 .

True or False: \vec{v} is an eigenvector of A^2 with eigenvalue 4.

(Remember, $A^2 = AA$; that is, it is the matrix A multiplied by itself.)

True

False

Solution: True.

Since \vec{v} is an eigenvector of A with eigenvalue -2 , we have $A\vec{v} = -2\vec{v}$.

Applying A again:

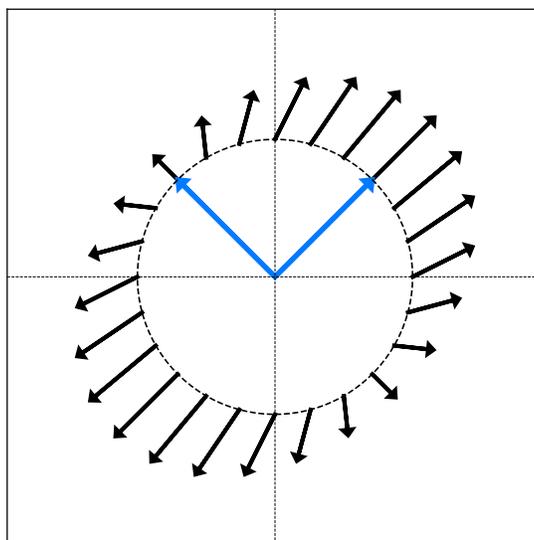
$$\begin{aligned} A^2\vec{v} &= A(A\vec{v}) \\ &= A(-2\vec{v}) \\ &= -2(A\vec{v}) \\ &= -2(-2\vec{v}) \\ &= 4\vec{v} \end{aligned}$$

So \vec{v} is an eigenvector of A^2 with eigenvalue $(-2)^2 = 4$.

In general, if \vec{v} is an eigenvector of A with eigenvalue λ , then \vec{v} is an eigenvector of A^k with eigenvalue λ^k .

Problem 3. (2 points)

The figure below shows a linear transformation \vec{f} applied to points on the unit circle. Each arrow shows the direction and relative magnitude of $\vec{f}(\vec{x})$ for a point \vec{x} on the circle.



The eigenvectors of the matrix A of this transformation are $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(1, -1)^T$. They are also plotted here as two arrows landing on the unit circle.

a) True or False: The matrix A is symmetric.

- True
- False

Solution: True.

Recall that a symmetric matrix has "axes of symmetry", and the action of the linear transformation is mirrored across those axes. Here, the eigenvectors $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ are the axes of symmetry, and the transformation is mirrored across them. Therefore, the matrix is symmetric.

b) True or False: The matrix A is diagonal.

- True
- False

Solution: False.

A diagonal matrix has the standard basis vectors $\hat{e}^{(1)}$ and $\hat{e}^{(2)}$ as its eigenvectors (i.e., eigenvectors along the x and y axes). Looking at the figure, the eigenvectors point at 45 angles, not along the axes. Therefore, the matrix is not diagonal.

c) Which eigenvector corresponds to the larger eigenvalue?

- $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$
- $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(1, -1)^T$

Solution: $\hat{u}^{(1)}$.

The eigenvalue tells us how much the transformation stretches vectors along that eigenvector direction. Looking at the figure, the arrows near the $\hat{u}^{(1)}$ direction (upper-right) are longer than the arrows near the $\hat{u}^{(2)}$ direction (lower-right). This means the transformation stretches more along $\hat{u}^{(1)}$, so it has the larger eigenvalue.

This was similar to Practice Problem 43.

Problem 4.

Let $\hat{u}^{(1)}, \hat{u}^{(2)}, \hat{u}^{(3)}$ be orthonormal eigenvectors of a linear transformation \vec{f} , with eigenvalues 3, -1 , and 2 respectively.

Suppose $\vec{x} = 3\hat{u}^{(1)} - 2\hat{u}^{(2)} + 4\hat{u}^{(3)}$.

What is $\vec{f}(\vec{x})$ in coordinates with respect to the eigenbasis $\mathcal{U} = \{\hat{u}^{(1)}, \hat{u}^{(2)}, \hat{u}^{(3)}\}$? That is, what is $[\vec{f}(\vec{x})]_{\mathcal{U}}$?

- $(9, -2, 8)^T$
- $(9, 2, 8)^T$
- $(3, -2, 4)^T$
- $(9, 2, -8)^T$

Solution: $[\vec{f}(\vec{x})]_{\mathcal{U}} = (9, 2, 8)^T$.

Using linearity and the eigenvector property:

$$\begin{aligned}\vec{f}(\vec{x}) &= \vec{f}(3\hat{u}^{(1)} - 2\hat{u}^{(2)} + 4\hat{u}^{(3)}) \\ &= 3\vec{f}(\hat{u}^{(1)}) - 2\vec{f}(\hat{u}^{(2)}) + 4\vec{f}(\hat{u}^{(3)}) \\ &= 3(3\hat{u}^{(1)}) - 2(-1 \cdot \hat{u}^{(2)}) + 4(2\hat{u}^{(3)}) \\ &= 9\hat{u}^{(1)} + 2\hat{u}^{(2)} + 8\hat{u}^{(3)}\end{aligned}$$

In the eigenbasis, this is $(9, 2, 8)^T$.

This was similar to Practice Problem 46.

Problem 5.

Let \vec{f} be a symmetric linear transformation in \mathbb{R}^3 with orthonormal eigenvectors $\hat{u}^{(1)}$, $\hat{u}^{(2)}$, $\hat{u}^{(3)}$ and corresponding eigenvalues 4, -7 , and 2.

What is the maximum value of $\|\vec{f}(\vec{x})\|$ over all unit vectors \vec{x} ?

7

Solution: 7.

The maximum of $\|\vec{f}(\vec{x})\|$ over unit vectors is achieved when \vec{x} is an eigenvector corresponding to the eigenvalue with the largest *absolute value*.

The eigenvalues are 4, -7 , and 2. Their absolute values are $|4| = 4$, $|-7| = 7$, and $|2| = 2$.

The largest absolute value is 7, corresponding to eigenvalue -7 . If $\vec{x} = \hat{u}^{(2)}$, then:

$$\|\vec{f}(\hat{u}^{(2)})\| = \|(-7)\hat{u}^{(2)}\| = |-7| \cdot \|\hat{u}^{(2)}\| = 7 \cdot 1 = 7$$

This was similar to Practice Problem 47.

Problem 6.

Let $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ be an orthonormal basis for \mathbb{R}^2 :

$$\begin{aligned}\hat{u}^{(1)} &= \frac{1}{\sqrt{2}}(1, 1)^T \\ \hat{u}^{(2)} &= \frac{1}{\sqrt{2}}(-1, 1)^T\end{aligned}$$

Let $\vec{x} = (1, 3)^T$ be a vector in the standard basis. What are the coordinates of \vec{x} in the basis $\mathcal{U} = \{\hat{u}^{(1)}, \hat{u}^{(2)}\}$? That is, what is $[\vec{x}]_{\mathcal{U}}$?

- $(\sqrt{2}, 2\sqrt{2})^T$
- $(2, 1)^T$
- $(2\sqrt{2}, \sqrt{2})^T$
- $(4, 2)^T$

Solution: $[\vec{x}]_{\mathcal{U}} = (2\sqrt{2}, \sqrt{2})^T$.

The change of basis matrix U has the basis vectors as its rows:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

We compute $[\vec{x}]_{\mathcal{U}} = U\vec{x}$:

$$\begin{aligned} [\vec{x}]_{\mathcal{U}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1+3 \\ -1+3 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} \end{aligned}$$

This was similar to Practice Problem 50.

Problem 7.

Let \vec{f} be a linear transformation with eigenvectors and eigenvalues:

$$\begin{aligned} \hat{u}^{(1)} &= \frac{1}{\sqrt{2}}(1, 1)^T & \lambda_1 &= 3 \\ \hat{u}^{(2)} &= \frac{1}{\sqrt{2}}(1, -1)^T & \lambda_2 &= -1 \end{aligned}$$

What is $\vec{f}(\vec{x})$ for $\vec{x} = (2, 0)^T$?

- $(4, 2)^T$
- $(6, -2)^T$
- $(3, -1)^T$
- $(2, 4)^T$

Solution: $\vec{f}(\vec{x}) = (2, 4)^T$.

We use the three-step approach: (1) find coordinates of \vec{x} in the eigenbasis, (2) apply the transformation in that basis, and (3) convert back to the standard basis.

The matrix of the transformation in the standard basis is $A = U^T \Lambda U$, where U has the eigenvectors as rows and Λ is the diagonal matrix of eigenvalues:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Computing A :

$$\begin{aligned} A &= U^T \Lambda U \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

Therefore:

$$\vec{f}(\vec{x}) = A\vec{x} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

This was similar to Practice Problem 53.

Problem 8.

Consider the matrix $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

The three eigenvectors of A are:

$$\begin{aligned} \hat{u}^{(1)} &= \frac{1}{\sqrt{3}}(1, 1, 1)^T \\ \hat{u}^{(2)} &= \frac{1}{\sqrt{2}}(1, -1, 0)^T \\ \hat{u}^{(3)} &= \frac{1}{\sqrt{6}}(1, 1, -2)^T \end{aligned}$$

What does the matrix A look like in the eigenbasis $\mathcal{U} = \{\hat{u}^{(1)}, \hat{u}^{(2)}, \hat{u}^{(3)}\}$?

- $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
- $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$
- $\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$
- $\begin{pmatrix} 6 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 6 \end{pmatrix}$

Solution: $A_{\mathcal{U}} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

In the eigenbasis, a matrix is always diagonal with the eigenvalues on the diagonal. We need to find the eigenvalues corresponding to each eigenvector.

To do this, we multiply A by each eigenvector and see what scalar multiple of the eigenvector we get

back. While we *could* multiply by the normalized eigenvectors as given, there is one small simplification we can make that will speed up our calculations: instead of multiplying A by $\frac{1}{\sqrt{3}}(1, 1, 1)^T$ to find the first eigenvalue, we can just multiply by $(1, 1, 1)^T$ and get the same result. This is because $(1, 1, 1)^T$ is also an eigenvector with the same eigenvalue.

For $(1, 1, 1)^T$:

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+1+1 \\ 1+4+1 \\ 1+1+4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \lambda_1 = 6$$

For $(1, -1, 0)^T$:

$$A \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4-1+0 \\ 1-4+0 \\ 1-1+0 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \lambda_2 = 3$$

For $(1, 1, -2)^T$:

$$A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4+1-2 \\ 1+4-2 \\ 1+1-8 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \Rightarrow \lambda_3 = 3$$

So the eigenvalues are 6, 3, and 3, giving the diagonal matrix shown.