DSC 140B - Quiz 02

Name: [ ] PID: [ ]

About the quizzes:

- Quizzes in DSC 140B are *optional* and graded pass/fail.
- A score of 70% or higher earns a "pass" and 1.5 credits toward your final grade.
- If you don't pass, no credits are earned, but it doesn't hurt your grade.
- You have 30 minutes to complete the quiz.
- At least one of the questions below will be on an exam (probably with slight changes, such as different numbers).
- Unfortunately, we can't answer clarifying questions during the quiz. If you think a question has a bug or is unclear, please let us known in a private post on Campuswire after the quiz, and we'll take it into account when grading.

Problem 1.

Let $\vec{x} = (3, -1, 4, 2, -5)^T \in \mathbb{R}^5$. What is $\vec{x} \cdot \hat{e}^{(4)}$?

2

Solution: 2.

The dot product $\vec{x} \cdot \hat{e}^{(4)}$ extracts the fourth component of $\vec{x}$, which is 2.

This was similar to Practice Problem 001.

Problem 2.

True or False: The following three vectors form an orthonormal basis for $\mathbb{R}^3$.

$$\vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

● True

○ False

Solution: True.

To form an orthonormal basis, the vectors must (1) all have unit length, and (2) be mutually orthogonal (each pair has dot product zero).

Checking unit length:

$$\|\vec{v}_1\| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = 1$$
$$\|\vec{v}_2\| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = 1$$
$$\|\vec{v}_3\| = \sqrt{0 + 1 + 0} = 1$$

Checking orthogonality:

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{2} + 0 - \frac{1}{2} = 0$$
$$\vec{v}_1 \cdot \vec{v}_3 = 0 + 0 + 0 = 0$$
$$\vec{v}_2 \cdot \vec{v}_3 = 0 + 0 + 0 = 0$$

Since all conditions are satisfied, the vectors form an orthonormal basis.

This was similar to Practice Problem 008.

Problem 3.

True or False: $\vec{f}(\vec{x}) = (x_1 + 4,\, x_2 - 1)^T$ is a linear transformation.

- ○ True
- ● False

Solution: False.

Any linear transformation must satisfy $\vec{f}(\vec{0}) = \vec{0}$. However:

$$\vec{f}(\vec{0}) = (0 + 4,\, 0 - 1)^T = (4, -1)^T \neq \vec{0}$$

Therefore, $\vec{f}$ is not a linear transformation.

This was similar to an example from Lecture 02, which asked whether $\vec{g}(\vec{x}) = (x_1 + 3, x_2)^T$ is a linear transformation. The answer was no, because $\vec{g}(\vec{0}) = (3, 0)^T \neq \vec{0}$.

Problem 4.

Suppose $\vec{f}$ is a linear transformation with:

$$\vec{f}(\hat{e}^{(1)}) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \vec{f}(\hat{e}^{(2)}) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Let $\vec{x} = (3, 2)^T$. Compute $\vec{f}(\vec{x})$.

- ○ $(8, -3)^T$
- ○ $(3, 2)^T$
- ○ $(5, 5)^T$
- ● $(8, 3)^T$

Solution: $\vec{f}(\vec{x}) = (8,3)^T$.

Since $\vec{f}$ is linear:

$$\begin{aligned}\vec{f}(\vec{x}) &= 3\vec{f}(\hat{e}^{(1)}) + 2\vec{f}(\hat{e}^{(2)}) \\ &= 3(2,-1)^T + 2(1,3)^T \\ &= (6,-3)^T + (2,6)^T \\ &= (8,3)^T\end{aligned}$$

This was similar to Practice Problem 013.

Problem 5.

Suppose $\vec{f}$ is a linear transformation with:

$$\vec{f}(\hat{e}^{(1)}) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \vec{f}(\hat{e}^{(2)}) = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

Which matrix A represents $\vec{f}$ with respect to the standard basis?

- ○ $\begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix}$
- ○ $\begin{pmatrix} -2 & 4 \\ 5 & 1 \end{pmatrix}$
- ● $\begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 5 \\ 4 & -2 \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix}$$

The columns of the matrix are $\vec{f}(\hat{e}^{(1)})$ and $\vec{f}(\hat{e}^{(2)})$.

This was similar to Practice Problem 020.

Problem 6.

Consider the linear transformation $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors by 180°.

What is the matrix A that represents $\vec{f}$ with respect to the standard basis?

- ● $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- ○ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- ○ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rotating by 180° sends each vector to its negative: $\vec{f}(\vec{x}) = -\vec{x}$. Therefore $\vec{f}(\hat{e}^{(1)}) = (-1, 0)^T$ and $\vec{f}(\hat{e}^{(2)}) = (0, -1)^T$, and the matrix has these vectors as its columns.

This was similar to Practice Problem 022.

Problem 7.

Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1,1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(1,-1)^T$ form an orthonormal basis $\mathcal{U}$ for $\mathbb{R}^2$.

Given $\vec{x} = (3,1)^T$, what is $[\vec{x}]_{\mathcal{U}}$? That is, what are the coordinates of $\vec{x}$ in the basis $\mathcal{U}$?

- ○ $(4,2)^T$
- ○ $(2,1)^T$
- ● $(2\sqrt{2}, \sqrt{2})^T$
- ○ $(\sqrt{2}, 2\sqrt{2})^T$

Solution: $[\vec{x}]_{\mathcal{U}} = (2\sqrt{2}, \sqrt{2})^T$.

To find the coordinates of $\vec{x}$ in the basis $\mathcal{U}$, we compute the dot product of $\vec{x}$ with each basis vector:

$$\begin{aligned} \vec{x} \cdot \hat{u}^{(1)} &= (3,1)^T \cdot \frac{1}{\sqrt{2}}(1,1)^T \\ &= \frac{1}{\sqrt{2}}(3+1) \\ &= \frac{4}{\sqrt{2}} \\ &= 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \vec{x} \cdot \hat{u}^{(2)} &= (3,1)^T \cdot \frac{1}{\sqrt{2}}(1,-1)^T \\ &= \frac{1}{\sqrt{2}}(3-1) \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2} \end{aligned}$$

Therefore, the coordinates of $\vec{x}$ in the basis $\mathcal{U}$ are:

$$[\vec{x}]_{\mathcal{U}} = (2\sqrt{2}, \sqrt{2})^T$$

This was similar to Practice Problem 002.

Problem 8.

Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1,1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(1,-1)^T$ form an orthonormal basis $\mathcal{U}$ for $\mathbb{R}^2$.

Suppose $[\vec{x}]_{\mathcal{U}} = (2\sqrt{2}, 3\sqrt{2})^T$. That is, the coordinates of $\vec{x}$ in the basis $\mathcal{U}$ are $(2\sqrt{2}, 3\sqrt{2})^T$.

What is $\vec{x}$ in the standard basis?

- ● $(5,-1)^T$
- ○ $(-1,5)^T$
- ○ $(5\sqrt{2}, -\sqrt{2})^T$
- ○ $(2\sqrt{2}, 3\sqrt{2})^T$

Solution: $\vec{x} = (5,-1)^T$.

To convert from basis $\mathcal{U}$ to the standard basis, we compute the linear combination of basis vectors:

$$\begin{aligned}\vec{x} &= [\vec{x}]_{\mathcal{U},1} \cdot \hat{u}^{(1)} + [\vec{x}]_{\mathcal{U},2} \cdot \hat{u}^{(2)} \\ &= 2\sqrt{2} \cdot \frac{1}{\sqrt{2}}(1,1)^T + 3\sqrt{2} \cdot \frac{1}{\sqrt{2}}(1,-1)^T \\ &= 2(1,1)^T + 3(1,-1)^T \\ &= (2,2)^T + (3,-3)^T \\ &= (5,-1)^T\end{aligned}$$

This was similar to Practice Problem 027.

Problem 9.

Let $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ be orthogonal unit vectors. We do not know their components, but we do know that they are orthonormal.

What is $(3\hat{u}^{(1)} + 2\hat{u}^{(2)}) \cdot (5\hat{u}^{(1)} + \hat{u}^{(2)})$?

- ● 17
- ○ 15
- ○ 30
- ○ 11

Solution: 17.

Since $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ are orthonormal, we have $\hat{u}^{(i)} \cdot \hat{u}^{(j)} = 1$ if $i = j$ and 0 otherwise.

Expanding the dot product:

$$\begin{aligned}(3\hat{u}^{(1)} + 2\hat{u}^{(2)}) \cdot (5\hat{u}^{(1)} + \hat{u}^{(2)}) &= 3 \cdot 5 \cdot (\hat{u}^{(1)} \cdot \hat{u}^{(1)}) + 3 \cdot 1 \cdot (\hat{u}^{(1)} \cdot \hat{u}^{(2)}) \\ &\quad + 2 \cdot 5 \cdot (\hat{u}^{(2)} \cdot \hat{u}^{(1)}) + 2 \cdot 1 \cdot (\hat{u}^{(2)} \cdot \hat{u}^{(2)}) \\ &= 15(1) + 3(0) + 10(0) + 2(1) \\ &= 15 + 2 \\ &= 17\end{aligned}$$

This comes from Fact #10 in the Math Review from Week 01. That fact reminded us that the dot product is *distributive*. That is, $(\vec{a} + \vec{b}) \cdot (\vec{x} + \vec{y}) = \vec{a} \cdot \vec{x} + \vec{a} \cdot \vec{y} + \vec{b} \cdot \vec{x} + \vec{b} \cdot \vec{y}$. Here we just applied that property and then used the orthonormality of the vectors to simplify.

Problem 10.

Suppose $\vec{v} = (3, 4)^T$.

Find a unit vector $\hat{u}$ such that $|\hat{u} \cdot \vec{v}|$ is maximized.

- ○ $(3/\sqrt{7}, 4/\sqrt{7})^T$
- ○ $(1/\sqrt{2}, 1/\sqrt{2})^T$
- ● $(3/5, 4/5)^T$
- ○ $(4/5, 3/5)^T$
- ○ $(3, 4)^T$
- ○ $(-4/5, 3/5)^T$

Solution: $\hat{u} = (3/5, 4/5)^T$ (or its negative).

To maximize $|\vec{v} \cdot \hat{u}|$, we need $|\cos(\theta)| = 1$, which happens when $\theta = 0°$ or $\theta = 180°$. In other words, $\hat{u}$ must be parallel to $\vec{v}$.

Normalizing $\vec{v}$: $\|\vec{v}\| = \sqrt{9 + 16} = 5$, so the unit vector is $(3, 4)^T/5 = (3/5, 4/5)^T$.

This was similar to a question on Quiz 01.