DSC 140B - Quiz 01

Name:

PID:

About the quizzes:

- Quizzes in DSC 140B are *optional* and graded pass/fail.
- A score of 70% or higher earns a “pass” and 1.5 credits toward your final grade.
- If you don’t pass, no credits are earned, but it doesn’t hurt your grade.
- You have 30 minutes to complete the quiz.
- At least one of the questions below will be on an exam (probably with slight changes, such as different numbers).
- Unfortunately, we can’t answer clarifying questions during the quiz. If you think a question has a bug or is unclear, please let us known in a private post on Campuswire after the quiz, and we’ll take it into account when grading.

Problem 1.

True or False: $\sum_{i=1}^{n} 6(x_i + 10) = (6\sum_{i=1}^{n} x_i) + 60n$.

● True

○ False

Solution: True. See Fact 2 and Fact 4 in the math review slides.

To see how you can prove these properties, see Fact 3.

Problem 2.

How should we interpret $\displaystyle\sum_{i=1}^{n} x_i + y_i$?

● $\sum_{i=1}^{n}(x_i + y_i)$

○ $(\sum_{i=1}^{n} x_i) + y_i$

Solution: It *has* to mean $\sum_{i=1}^{n}(x_i + y_i)$, because $(\sum_{i=1}^{n} x_i) + y_i$ does not make notational sense!

Just like in programming, i is “unbound” in the second choice, so y_i is not defined!

Problem 3.

Compute $(1,4,3)^T + (2,0,1)^T$.

$(3,4,4)^T$

Solution: $(1,4,3)^T + (2,0,1)^T = (1+2, 4+0, 3+1)^T = (3,4,4)^T$. From Fact 6.

Problem 4.

Compute $4(1,4,3)^T$.

$(4,16,12)^T$

Solution: $4(1,4,3)^T = (4\cdot 1, 4\cdot 4, 4\cdot 3)^T = (4,16,12)^T$. From Fact 7.

Problem 5.

Compute $(1,4,3)^T \cdot (2,0,1)^T$. Here, $\cdot$ denotes the dot product.

5

Solution: $(1,4,3)^T \cdot (2,0,1)^T = 1\cdot 2 + 4\cdot 0 + 3\cdot 1 = 5$. From Fact 8.

Problem 6.

Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal to one another (the angle between them is $90°$). What is $\vec{u}\cdot\vec{v}$?

0

Solution: Zero.

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(90°)$. Since $\cos(90°) = 0$, the dot product is zero. From Fact 9.

Problem 7.

$\vec{u} = (1,2,3)^T$. What is the length of $\vec{u}$? (You can leave your answer unsimplified.)

$\sqrt{14}$

Solution: $\|\vec{u}\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$. From Fact 5.

Problem 8.

Suppose $\vec{v} = (3,3)^T$.

a) Find a unit vector $\vec{u}^{(1)}$ such that $\vec{u}^{(1)} \cdot \vec{v} = 0$. (You can leave your answer unsimplified.)

$(1,-1)^T/\sqrt{2}$

Solution: $\vec{u}^{(1)} = (1,-1)^T/\sqrt{2}$.

This comes from Fact 9. That fact tells us that $\vec{v} \cdot \vec{u}^{(1)} = \|v\|\|\vec{u}^{(1)}\|\cos(\theta)$. $\vec{u}^{(1)}$ is a unit vector, so $\|\vec{u}^{(1)}\| = 1$. Likewise, $\|\vec{v}\|$ is fixed – we can't change it. So to make the dot product zero, we need to make $\cos(\theta) = 0$, which happens when $\theta = 90°$, meaning $\vec{u}^{(1)}$ is orthogonal to $\vec{v}$. You can check that $(1,-1)^T/\sqrt{2}$ is a unit vector, and it's orthogonal to $\vec{v}$.

b) Find a unit vector $\vec{u}^{(2)}$ such that $|\vec{u}^{(2)} \cdot \vec{v}|$ is maximized. (You can leave your answer unsimplified.)

$(1,1)^T/\sqrt{2}$

Solution: $\vec{u}^{(2)} = (1,1)^T/\sqrt{2}$.

To maximize $|\vec{v} \cdot \vec{u}^{(2)}|$, we need to make $\cos(\theta) = 1$, which happens when $\theta = 0°$. In other words, when $\vec{u}^{(2)}$ is parallel to $\vec{v}$.

Problem 9.

Which of these is another expression for the norm of $\vec{u}$ (that is, $\|\vec{u}\|$)?

- ○ $\vec{u} \cdot \vec{u}$
- ○ $\sqrt{\vec{u}^2}$
- ● $\sqrt{\vec{u} \cdot \vec{u}}$
- ○ $\vec{u}^2$

Solution: This also comes from Fact 5 and/or Fact 9.

Using the coordinate definition of the dot product, we see that the norm of $\vec{u}$ is $\sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_d^2}$, which we recognize as the Euclidean norm.

Another way to see this is to use the geometric definition of the dot product: $\vec{u} \cdot \vec{u} = \|\vec{u}\|\|\vec{u}\|\cos(0°) = \|\vec{u}\|^2$, where we have said that the angle between $\vec{u}$ and itself is $0°$. It follows that $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$.

Problem 10.

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, and let α, β be scalars.

True or False: $\vec{u} \cdot (\alpha\vec{v} + \beta\vec{w}) = \alpha\vec{u} \cdot \vec{v} + \beta\vec{u} \cdot \vec{w}$.

- ● True
- ○ False

Solution: True. This is the "linear" property of the dot product. See Fact 10.

Problem 11.

Let A, B, C, X be matrices of appropriate dimensions. True or False: $X(AB + C)^T = XB^T A^T + XC^T$.

- ● True
- ○ False

Solution: True. Matrix-matrix multiplication is distributive (see Fact 16).

Also, the transpose of a product is the product of the transposes in reverse order. That is, $(AB)^T = B^T A^T$. See Fact 17.

Problem 12.

Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{pmatrix},$$

and let $\vec{x} = (0, 1, 0, 2, 0)^T$.

What is $A\vec{x}$?

$(10, 25, 40)^T$

Solution: $(2, 7, 12)^T + (8, 18, 28)^T = (10, 25, 40)^T$.

Rather than carrying out the full matrix-vector multiplication, you might want to instead use Fact 12, which says that $A\vec{x}$ is a linear combination of the columns of A, where the coefficients are the entries of $\vec{x}$. So here, the result should be one copy of the second column of A and two copies of the fourth column of A.

Problem 13.

Let A, B and C be matrices of appropriate dimensions.

True or False: $ABC = CBA$.

- ○ True
- ● False

Solution: False. Matrix multiplication is not commutative. See Fact 16.

Problem 14.

Let $\vec{x} \in \mathbb{R}^d$ and let A be a $d \times d$ matrix. What type of object is $\vec{x}^T A \vec{x}$?

- ● A scalar
- ○ A vector in $\mathbb{R}^d$
- ○ A vector in $\mathbb{R}^n$
- ○ A matrix in $\mathbb{R}^{d \times d}$
- ○ A matrix in $\mathbb{R}^{n \times n}$

- ◯ A matrix in $\mathbb{R}^{n \times d}$

Solution: A scalar. See Fact 20.

Namely, $A\vec{x}$ is a d-dimensional vector, which we can think of as a $d \times 1$ matrix. Then $\vec{x}^T A\vec{x}$ is a $1 \times d$ matrix times a $d \times 1$ matrix, which is a 1×1 matrix, or a scalar.

Problem 15.

Let $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ be d-dimensional vectors. What type of object is:

$$\frac{1}{n}\sum_{i=1}^{n} \vec{x}^{(i)}(\vec{x}^{(i)})^T$$

- ◯ A scalar
- ◯ A vector in $\mathbb{R}^d$
- ◯ A vector in $\mathbb{R}^n$
- ● A matrix in $\mathbb{R}^{d \times d}$
- ◯ A matrix in $\mathbb{R}^{n \times n}$
- ◯ A matrix in $\mathbb{R}^{n \times d}$

Solution: A $d \times d$ matrix. See Fact 20.

$\vec{x}^{(i)}$ is a column vector, which we can think of as a $d \times 1$ matrix. On the other hand, $(\vec{x}^{(i)})^T$ is a row vector, which we can think of as a $1 \times d$ matrix. So the product $\vec{x}^{(i)}(\vec{x}^{(i)})^T$ is $(d \times 1)(1 \times d)$, which is a $d \times d$ matrix.

So each term in the sum is a $d \times d$ matrix, and the sum of n such terms is also a $d \times d$ matrix. Dividing by n doesn't change the type of object.