DSC 140B

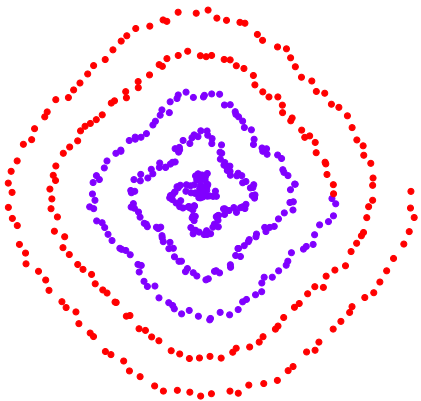
Representation Learning

Lecture 08 | Part 1

Recap

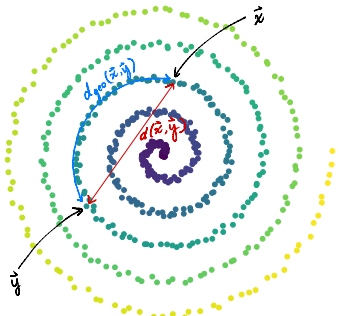
Last Time

- PCA is a **linear** dimension reduction technique.
  - Works well when data lies near a **linear** subspace.
- What if data lies on a **nonlinear** manifold?



Euclidean vs. Geodesic Distance

- **Euclidean distance**: the “straight-line” distance
- **Geodesic distance**: the distance along the manifold



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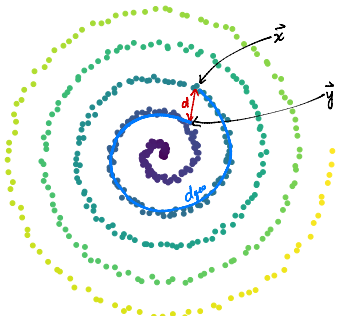
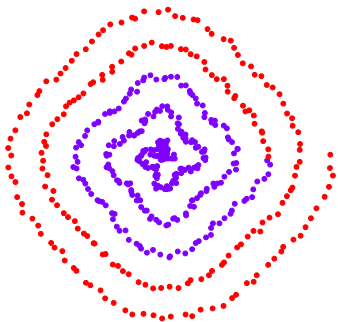


Non-Linear Dimensionality Reduction

- **Goal**: Map points in $\mathbb{R}^d$ to $\mathbb{R}^k$
- **Such that**: if $\vec{x}$ and $\vec{y}$ are close in **geodesic** distance in $\mathbb{R}^d$, they are close in **Euclidean** distance in $\mathbb{R}^k$

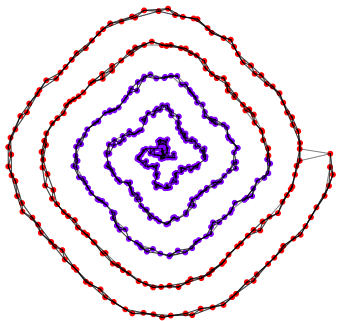
Computing Geodesic Distance

- But how do we **compute** geodesic distance?



Computing Geodesic Distance

- But how do we **compute** geodesic distance?
- **Idea**: build a **similarity graph**.
- Now, geodesic distance ≈ shortest path distance on graph.



The Plan

- **Before**: Map points close in geodesic distance to points close in $\mathbb{R}^k$.

- **Now**:
  - Convert points to a similarity graph.
  - Map similar nodes to points close in $\mathbb{R}^k$.

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Representation Learning

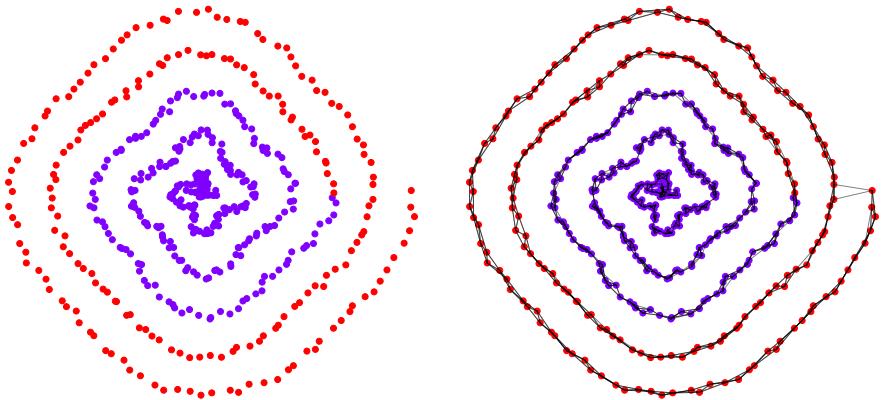
Lecture 08 | Part 2

From Points to Graphs

Goal

- **Given**: a set of n points in $\mathbb{R}^d$
  - We think they're on a (non-linear) manifold
- **Goal**: construct a **similarity graph** where:
  - Each point corresponds to a node
  - Edges connect points close in **geodesic distance**

Example



Three Approaches

- 1) Epsilon neighbors graph
- 2) k-Nearest neighbor graph
- 3) fully connected graph with similarity function

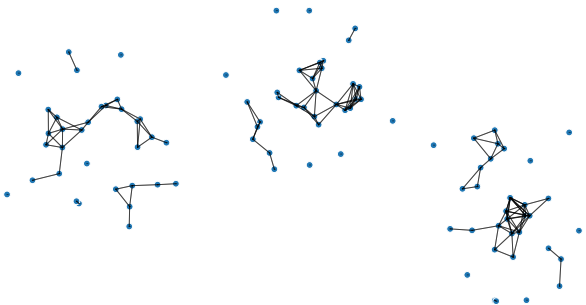
Epsilon Neighbors Graph

- **Input:** vectors $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$, a number ε
- Create a graph with one node i per point $\vec{x}^{(i)}$
- Add edge between nodes i and j if $\|\vec{x}^{(i)} - \vec{x}^{(j)}\| \leq \varepsilon$
- **Result**: **unweighted** graph

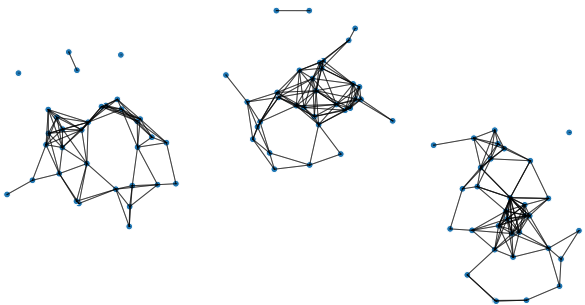
Exercise

What will the graph look like when ε is small? What about when it is large?

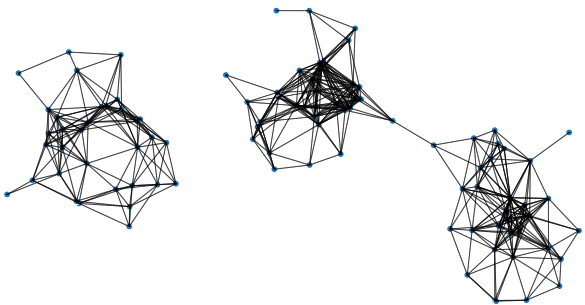
Epsilon Neighbors Graph



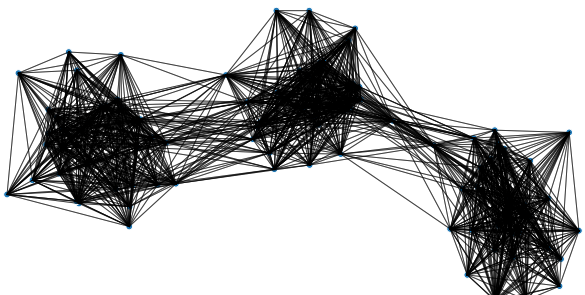
Epsilon Neighbors Graph



Epsilon Neighbors Graph



Epsilon Neighbors Graph



Note

- We've drawn these graphs by placing nodes at the same position as the point they represent
- But a graph's nodes can be drawn anywhere.
- **Key Point**: the graph has no information about where the points were; just which were **close**.

Epsilon Neighbors: Pseudocode

```
# assume the data is in X
n = len(X)
adj = np.zeros_like(X)
for i in range(n):
    for j in range(n):
        if distance(X[i], X[j]) <= epsilon:
            adj[i, j] = 1
```

Picking ε

- If ε is **too small**, graph is underconnected
- If ε is **too large**, graph is overconnected
  - Not capturing the manifold structure
- If you cannot visualize, test different values and pick one that gives good results

With scikit-learn

```
import sklearn.neighbors
adj = sklearn.neighbors.radius_neighbors_graph(
        X,
        radius=...
    )
```

k-Neighbors Graph

- **Input**: vectors $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$, a number k
- Create a graph with one node i per point $\vec{x}^{(i)}$
- Add edge between each node i and its k closest neighbors
- **Result**: **unweighted** graph

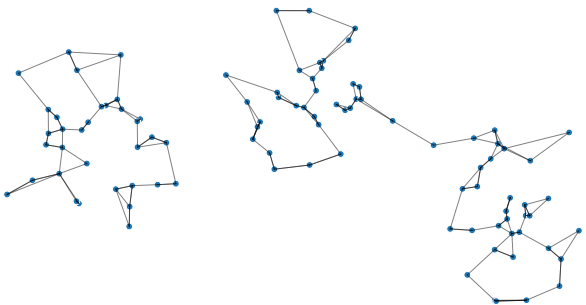
k-Neighbors: Pseudocode

```
# assume the data is in X
n = len(X)
adj = np.zeros_like(X)
for i in range(n):
    for j in k_closest_neighbors(X, i):
        adj[i, j] = 1
```

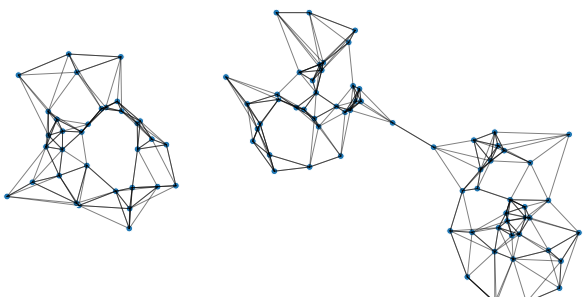
Exercise

True or False: Is it possible for a k-neighbors graph to be disconnected?

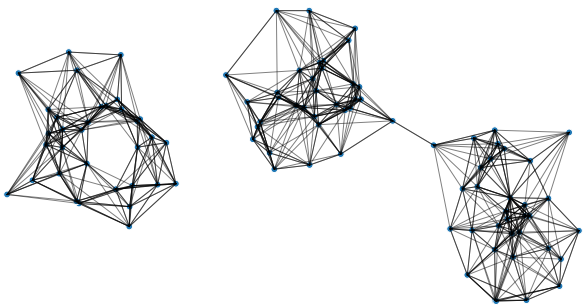
k-Neighbors Graph



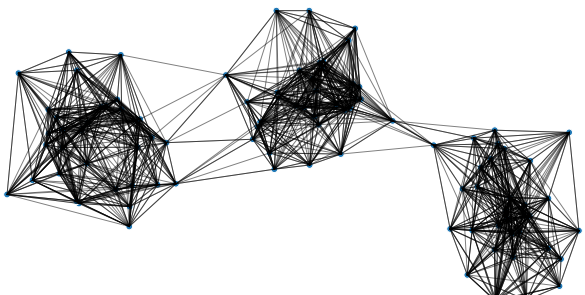
k-Neighbors Graph



k-Neighbors Graph



k-Neighbors Graph



With scikit-learn

```
import sklearn.neighbors
adj = sklearn.neighbors.kneighbors_graph(
        X,
        n_neighbors=...
    )
```

Fully Connected Graph

- **Input**: vectors $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$, a **similarity function** h
- Create a graph with one node i per point $\vec{x}^{(i)}$
- Add edge between every pair of nodes. Assign weight of $h(\vec{x}^{(i)}, \vec{x}^{(j)})$
- **Result**: **weighted** graph

Similarity Functions

- A **similarity function** $h(\vec{x}, \vec{y})$ measures how "similar" two points are

- A good similarity function should satisfy:
  - Output is positive: $h(\vec{x}, \vec{y}) \geq 0$
  - Symmetry: $h(\vec{x}, \vec{y}) = h(\vec{y}, \vec{x})$
  - Larger values = more similar

Gaussian Similarity

- A common similarity function: Gaussian[1]
- Must choose σ appropriately:

$$h(\vec{x}, \vec{y}) = e^{-\|\vec{x}-\vec{y}\|^2/\sigma^2}$$

[1] Note that this isn't a normalized Gaussian pdf, but we still call it "Gaussian".

Fully Connected: Pseudocode

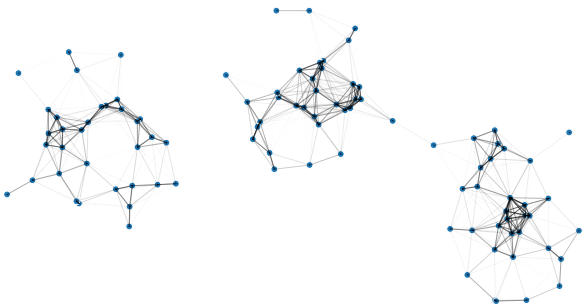
```
def h(x, y):
    dist = np.linalg.norm(x, y)
    return np.exp(-dist**2 / sigma**2)

# assume the data is in X
n = len(X)
w = np.ones_like(X)
for i in range(n):
    for j in range(n):
        w[i, j] = h(X[i], X[j])
```

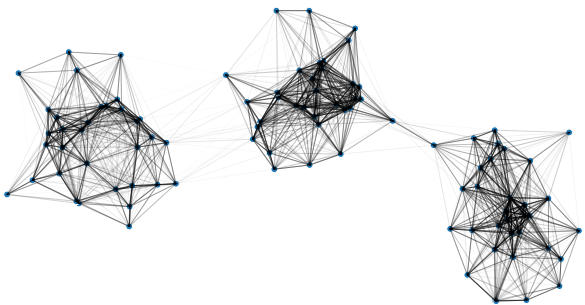
With SciPy

```
distances = scipy.spatial.distance_matrix(X, X)
w = np.exp(-distances**2 / sigma**2)
```

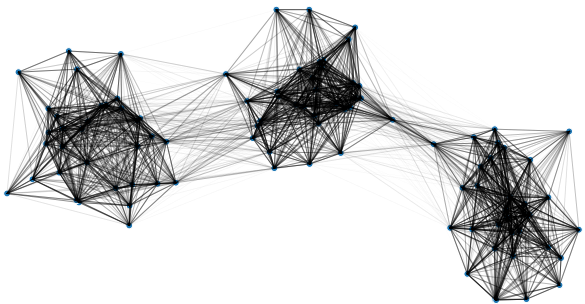
Gaussian Similarity



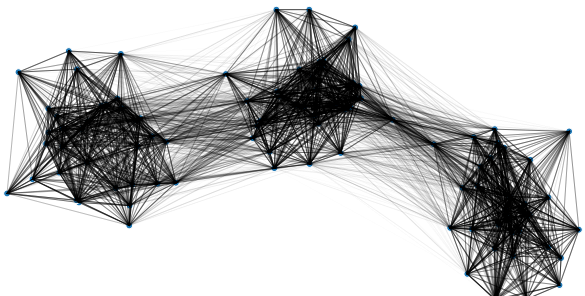
Gaussian Similarity



Gaussian Similarity



Gaussian Similarity



Next Step

- **Overall goal:** map points that are close in geodesic distance in $\mathbb{R}^d$ to points that are close in Euclidean distance in $\mathbb{R}^k$
- **First step**: convert points to similarity graph
- **Next step**: map similar nodes to points close in $\mathbb{R}^k$

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Representation Learning

Lecture 08 | Part 3

Embedding Similarities

Similar Netflix Users

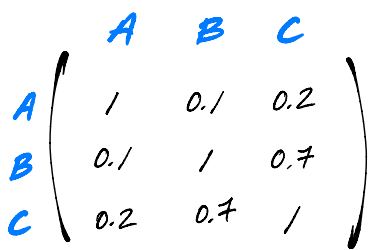
- Suppose you are a data scientist at Netflix
- You're given an $n \times n$ **similarity matrix** W of users
  - entry (i, j) tells you how *similar* user i and user j are
  - 1 means "very similar", 0 means "not at all"
- **Goal**: visualize to find patterns

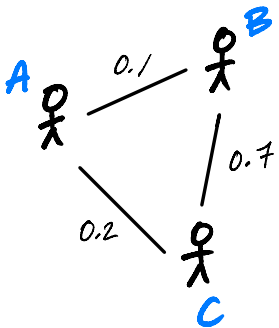
Idea

- We like scatter plots. Can we make one?
- Users are **not** vectors / points!
- They are **nodes** in a **similarity graph**

Similarity Graphs

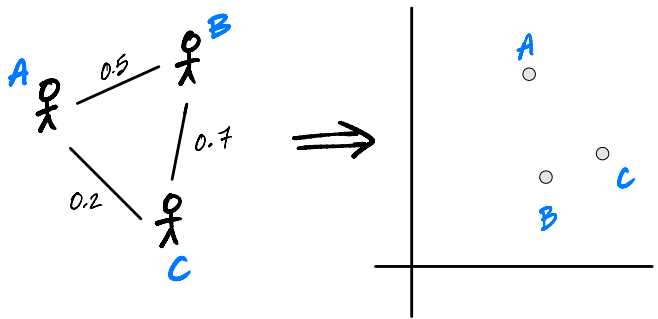
- Similarity matrices can be thought of as weighted graphs, and *vice versa*.





Goal

- **Embed** nodes of a similarity graph as points.
- Similar nodes should map to nearby points.



Today

- We will design a graph embedding approach:
  - **Spectral embeddings** via **Laplacian eigenmaps**

More Formally

- **Given**:
  - A **similarity graph** with n nodes
  - a number of dimensions, k

- **Compute**: an **embedding** of the n nodes into $\mathbb{R}^k$ so that similar objects are placed nearby

To Start

- **Given**:
  - A **similarity graph** with n nodes
  - a number of dimensions, k

- **Compute**: an **embedding** of the n nodes into $\mathbb{R}^1$ so that similar objects are placed nearby

Vectors as Embeddings into $\mathbb{R}^1$

- Suppose we have n nodes (objects) to embed
- Assume they are numbered 1, 2, ..., n
- Let $f_1, f_2, \ldots, f_n \in \mathbb{R}$ be the embeddings
- We can pack them all into a vector: $\vec{f}$.
- **Goal**: find a good set of embeddings, $\vec{f}$.

Example

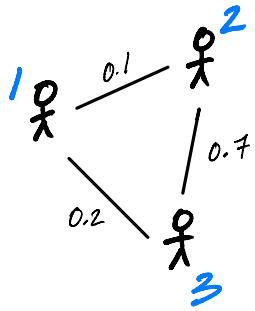
$\vec{f} = (1, 3, 2, -4)^T$

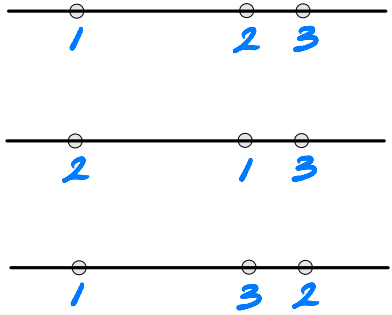
An Optimization Problem

- We'll turn it into an optimization problem:
- **Step 1**: Design a cost function quantifying how good a particular embedding $\vec{f}$ is
- **Step 2**: Minimize the cost

Exercise

Which of the following embeddings is intuitively the best, given the similarity graph below?





Cost Function for Embeddings

- **Idea**: cost is low if similar points are close
- Here is one approach:

$$\text{Cost}(\vec{f}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} (f_i - f_j)^2$$

- where w_{ij} is the weight between i and j.

Interpreting the Cost

$$\text{Cost}(\vec{f}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(f_i - f_j)^2$$

- If $w_{ij} \approx 0$, that pair can be placed very far apart without increasing cost
- If $w_{ij} \approx 1$, the pair should be placed close together in order to have small cost.

Exercise

Do you see a problem with the cost function?

$$\text{Cost}(\vec{f}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(f_i - f_j)^2$$

Hint: what embedding $\vec{f}$ minimizes it?

Problem

- The cost is **always** minimized by taking $\vec{f} = 0$.
- This is a "**trivial**" solution. Not useful.
- **Fix**: require $\|\vec{f}\| = 1$
  - Really, any number would work. 1 is convenient.

Exercise

Do you see **another** problem with the cost function, even if we require $\vec{f}$ to be a unit vector?

$$\text{Cost}(\vec{f}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(f_i - f_j)^2$$

Hint: what other choice of $\vec{f}$ will **always** make this zero?

Problem

- The cost is **always** minimized by taking $\vec{f} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$.
- This is a "**trivial**" solution. Again, not useful.
- **Fix**: require $\vec{f}$ to be orthogonal to $(1, 1, \ldots, 1)^T$.
  - Written: $\vec{f} \perp (1, 1, \ldots, 1)^T$
  - Ensures that solution is not close to trivial solution
  - Might seem strange, but it will work!

The New Optimization Problem

- **Given**: an $n \times n$ similarity matrix W
- **Compute**: embedding vector $\vec{f}$ minimizing

$$\text{Cost}(\vec{f}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} (f_i - f_j)^2$$

subject to $\|\vec{f}\| = 1$ and $\vec{f} \perp (1, 1, \ldots, 1)^T$

How?

- This looks difficult.
- Let’s write it in matrix form.
- We’ll see that it is actually (hopefully) familiar.

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Representation Learning

Lecture 08 | Part 4

The Graph Laplacian

The Problem

- **Compute**: embedding vector $\vec{f}$ minimizing

$$\text{Cost}(\vec{f}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(f_i - f_j)^2$$

  subject to $\|\vec{f}\| = 1$ and $\vec{f} \perp (1, 1, \ldots, 1)^T$

- Now: write the cost function as a matrix expression.

The Degree Matrix

- **Recall**: in an unweighted graph, the degree of node i equals number of neighbors.
- Equivalently (where A is the adjacency matrix):

$$\text{degree}(i) = \sum_{j=1}^{n} A_{ij}$$

- Since $A_{ij} = 1$ only if j is a neighbor of i

The Degree Matrix

- In a weighted graph, define **degree** of node i similarly:

$$\text{degree}(i) = \sum_{j=1}^{n} w_{ij}$$

- That is, it is the total weight of all neighbors.

The Degree Matrix

- The **degree matrix** D of a weighted graph is the diagonal matrix where entry (i, i) is given by:

$$\begin{aligned} d_{ii} &= \text{degree}(i) \\ &= \sum_{j=1}^{n} w_{ij} \end{aligned}$$

The Graph Laplacian

- Define $L = D - W$
  - D is the degree matrix
  - W is the similarity matrix (weighted adjacency)
- L is called the **Graph Laplacian** matrix.
- It is a very useful object

Very Important Fact

- **Claim**:

$$\text{Cost}(\vec{f}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(f_i - f_j)^2 = \vec{f}^T L \vec{f}$$

- **Proof**: expand both sides.

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Representation Learning

Lecture 08 | Part 5

Solving the Optimization Problem

A New Formulation

- **Given**: an $n \times n$ similarity matrix W
- **Compute**: embedding vector $\vec{f}$ **minimizing**

$$\text{Cost}(\vec{f}) = \vec{f}^T L \vec{f}$$

subject to $\|\vec{f}\| = 1$ and $\vec{f} \perp (1, 1, \ldots, 1)^T$

- This might sound familiar...

Recall: PCA

- **Given**: a $d \times d$ covariance matrix C
- **Find**: vector $\vec{u}$ **maximizing** the variance in the direction of $\vec{u}$:

$$\vec{u}^T C \vec{u}$$

subject to $\|\vec{u}\| = 1$.

- **Solution**: take $\vec{u}$ = top eigenvector of C

A New Formulation

- Forget about orthogonality constraint for now.
- **Compute**: embedding vector $\vec{f}$ **minimizing**

$$\text{Cost}(\vec{f}) = \vec{f}^T L \vec{f}$$

subject to $\|\vec{f}\| = 1$.

- **Solution**: the *bottom* eigenvector of L.
  - That is, eigenvector with smallest eigenvalue.

Claim

- The bottom eigenvector is $\vec{f} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$
- It has associated eigenvalue of 0.
- That is, $L\vec{f} = 0\vec{f} = \vec{0}$

Spectral[2] Theorem

Theorem
If A is a symmetric matrix, eigenvectors of A with distinct eigenvalues are orthogonal to one another.

[2] "Spectral" not in the sense of specters (ghosts), but because the eigenvalues of a transformation form the "spectrum"

The Fix

- **Remember**: we wanted $\vec{f}$ to be orthogonal to $\frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$.
  - i.e., should be orthogonal to bottom eigenvector of L.
- **Fix**: take $\vec{f}$ to the be eigenvector of L with with smallest eigenvalue $\neq 0$.
- Will be $\perp \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$ by the **spectral theorem**.

Spectral Embeddings: Problem

- **Given**: **similarity graph** with n nodes
- **Compute**: an **embedding** of the n points into $\mathbb{R}^1$ so that similar objects are placed nearby
- **Formally**: find embedding vector $\vec{f}$ **minimizing**

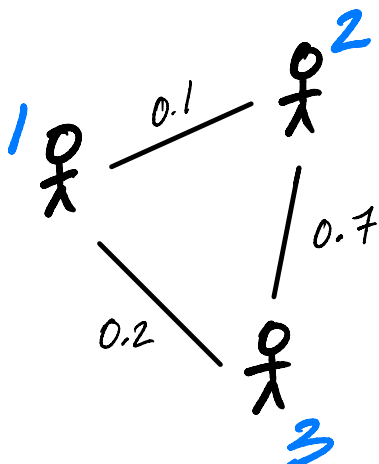
$$\text{Cost}(\vec{f}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(f_i - f_j)^2 = \vec{f}^T L \vec{f}$$

subject to $\|\vec{f}\| = 1$ and $\vec{f} \perp (1, 1, \ldots, 1)^T$

Spectral Embeddings: Solution

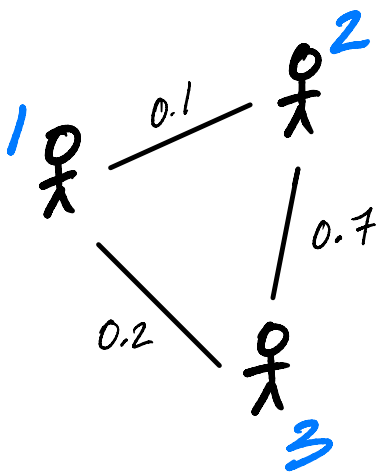
- Form the **graph Laplacian** matrix, $L = D - W$
- Choose $\vec{f}$ be an eigenvector of L with smallest eigenvalue > 0
- This is the embedding!

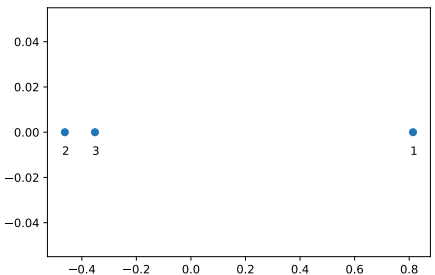
Example




```
W = np.array([
    [1, 0.1, 0.2],
    [0.1, 1, 0.7],
    [0.2, 0.7, 1]
])

D = np.diag(W.sum(axis=1))
L = D - W

vals, vecs = np.linalg.eigh(L)

f = vecs[:,1]
```

Example





Embedding into $\mathbb{R}^k$

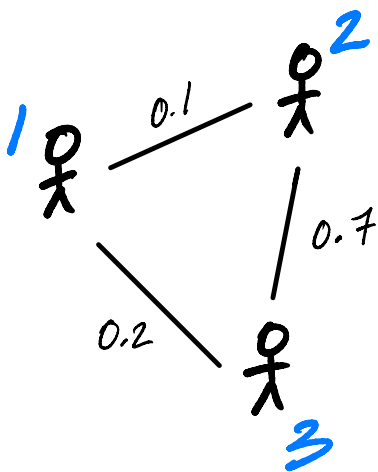
- This embeds nodes into $\mathbb{R}^1$.
- What about embedding into $\mathbb{R}^k$?
- Natural extension: find bottom k eigenvectors with eigenvalues > 0

New Coordinates

- With k eigenvectors $\vec{f}^{(1)}, \vec{f}^{(2)}, \ldots, \vec{f}^{(k)}$, each node is mapped to a point in $\mathbb{R}^k$.

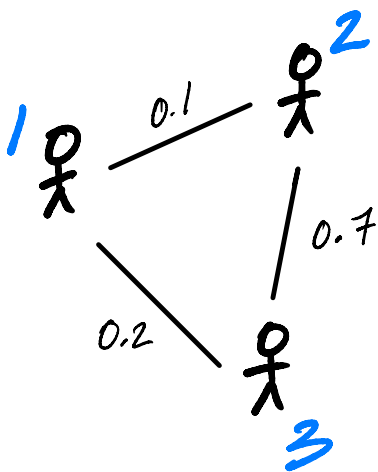
- Consider node i.
  - First new coordinate is $\vec{f}_i^{(1)}$.
  - Second new coordinate is $\vec{f}_i^{(2)}$.
  - Third new coordinate is $\vec{f}_i^{(3)}$.
  - $\vdots$

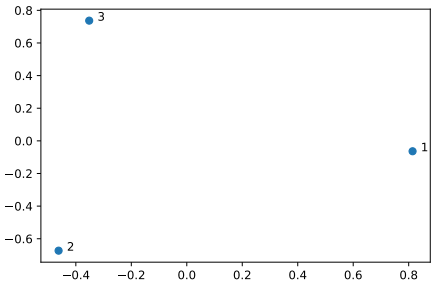
Example




```
W = np.array([
    [1, 0.1, 0.2],
    [0.1, 1, 0.7],
    [0.2, 0.7, 1]
])

D = np.diag(W.sum(axis=1))
L = D - W

vals, vecs = np.linalg.eigh(L)

# take two eigenvectors
# to map to R^2
f = vecs[:,1:3]
```

Example





Laplacian Eigenmaps

- This approach is part of the method of "**Laplacian eigenmaps**"
- Introduced by Mikhail Belkin[3] and Partha Niyogi
- It is a type of **spectral embedding**

[3] Now at HDSI

A Practical Issue

- The Laplacian is often **normalized**:

$$L_{\text{norm}} = D^{-1/2} L D^{-1/2}$$

  where $D^{-1/2}$ is the diagonal matrix whose ith diagonal entry is $1/\sqrt{d_{ii}}$.

- Proceed by finding the eigenvectors of L_{norm}.

In Summary

- We can **embed** a similarity graph's nodes into $\mathbb{R}^k$ using the eigenvectors of the graph Laplacian
- Yet another instance where eigenvectors are solution to optimization problem
- Next time: using this for dimensionality reduction

DSC 140B

Representation Learning

Lecture 08 | Part 6

Laplacian Eigenmaps

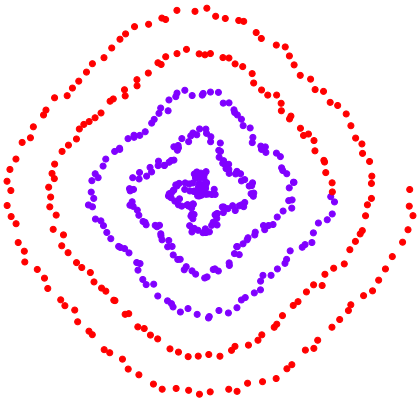
Problem: Non-linear Dimensionality Reduction

- **Given**: points in $\mathbb{R}^d$, target dimension k
- **Goal**: **embed** the points in $\mathbb{R}^k$ so that points that were close in geodesic distance are close after

Idea

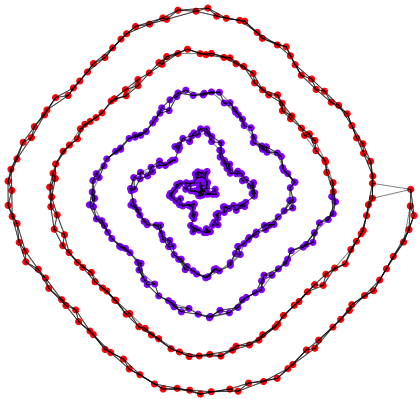
- Build a similarity graph from points in $\mathbb{R}^d$
  - epsilon neighbors, k-neighbors, or fully connected
- Embed the similarity graph in $\mathbb{R}^k$ using eigenvectors of graph Laplacian

Example 1: Spiral



Example 1: Spiral

- Build a k-neighbors graph.
- Note: follows the 1-d shape of the data.



Example 1: Spectral Embedding

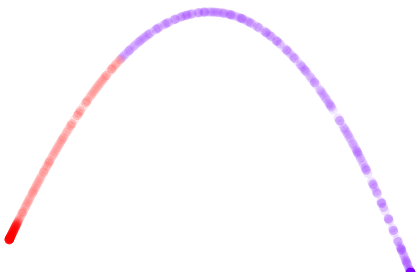
- Let W be the weight matrix (k-neighbor adjacency matrix)
- Compute $L = D - W$
- Compute bottom k non-zero eigenvectors of L, use as embedding

Example 1: Spiral

- Embedding into $\mathbb{R}^1$

Example 1: Spiral

- Embedding into $\mathbb{R}^2$



Example 1: Spiral

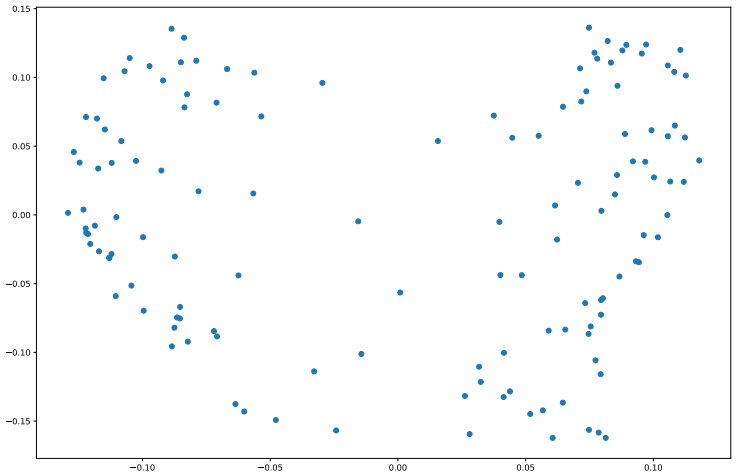
```
import sklearn.neighbors
import sklearn.manifold
W = sklearn.neighbors.kneighbors_graph(
        X, n_neighbors=4
)
embedding = sklearn.manifold.spectral_embedding(
        W, n_components=2
)
```

Example 2: Face Pose

Example 2: Face Pose

- Construct fully-connected similarity graph with Gaussian similarity
- Embed with Laplacian eigenmaps

Example 2: Face Pose



Example 2: Face Pose