DSC 140B

Representation Learning

Lecture 02 | Part 1

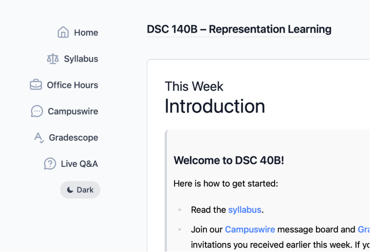
News

News

- **Homework 01 released**, due next Wednesday.
  - Optional!
  - Remember: must be handwritten.
- **Quiz 01 tonight**, 8pm (in this room).
  - Optional!

Attendance

- To earn credit for lecture attendance, you'll need to respond to (most of) the in-class polls.
- We'll use the "Live Q&A" on the course page:

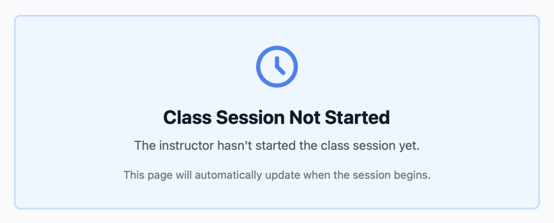


Important!

- To use the Live Q&A, you need to be on UCSD wifi.
  - If you're not, the page won't load.

Setup

- Go ahead and open the Live Q&A page now.
  - Remember, it is linked at `dsc140b.com`.
- You'll be asked for your PID. Enter it.
- You should then see:[1]



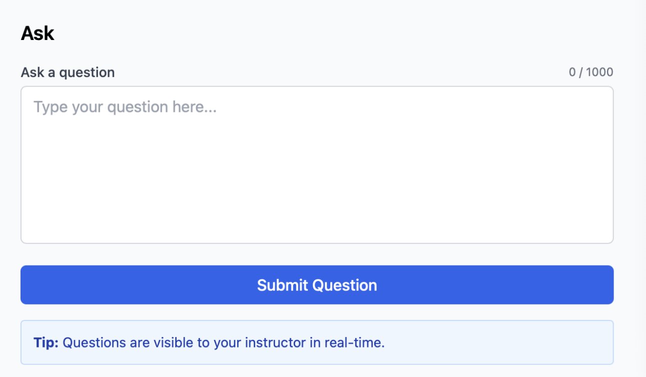
[1]If you couldn't get it to work, let me know after class.

Exercise

Do you plan on taking the quiz tonight?

- True = Yes
- False = No

By the way...

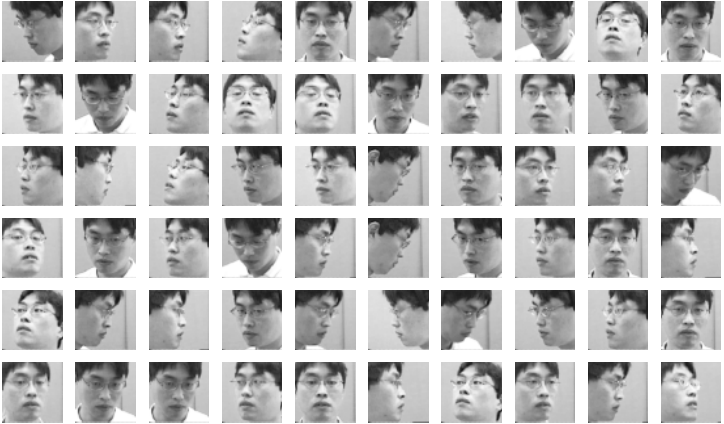

Ask
Ask a question
0 / 1000
Type your question here...
Submit Question
Tip: Questions are visible to your instructor in real-time.

DSC 140B
Representation Learning

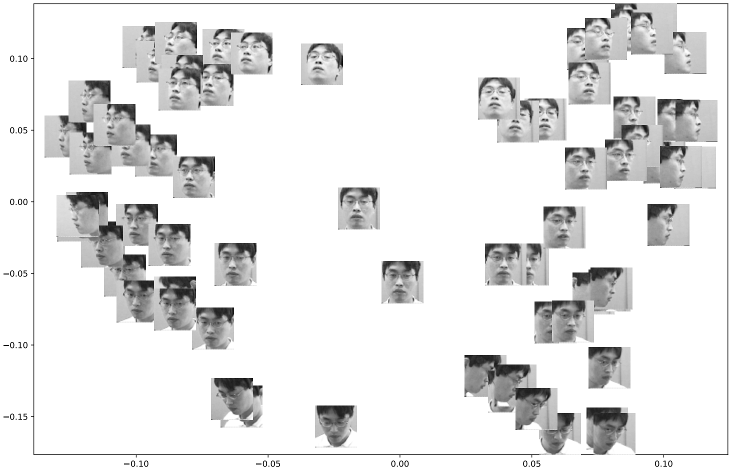
Lecture 02 | Part 2

Why Linear Algebra?

Last Time



Last Time



Dimensionality Reduction

- This is an example of **dimensionality reduction**:
  - Input: vectors in $\mathbb{R}^{10,000}$.
  - Output: vectors in $\mathbb{R}^2$.

- The method which produced this result is called **Laplacian Eigenmaps**.

- How does it work?

A Preview of Laplacian Eigenmaps

To reduce dimensionality from d to d':

1. Create an undirected **similarity graph** G
   - Each vector in $\mathbb{R}^d$ becomes a node in the graph.
   - Make edge (u, v) if u and v are "close"
2. Form the **graph Laplacian matrix**, L:
   - Let A be the adjacency matrix, D be the degree matrix.
   - Define the graph Laplacian matrix, $L = D - A$.
3. Compute d' **eigenvectors** of L.
   - Each eigenvector gives one new feature.

Why eigenvectors?

- We will cover Laplacian Eigenmaps in much greater detail.

- For now: why do eigenvectors appear here?
  - What are eigenvectors?
  - How are they useful?
  - Why is linear algebra important in ML?

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Representation Learning

Lecture 02 | Part 3

Coordinate Vectors

Coordinate Vectors

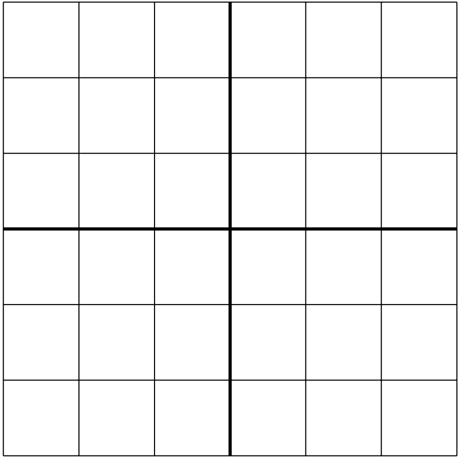
- We can write a vector $\vec{x} \in \mathbb{R}^d$ as a **coordinate vector**:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$$

Example

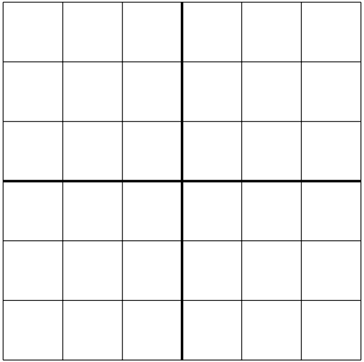
$$\vec{x} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\vec{y} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$



Standard Basis

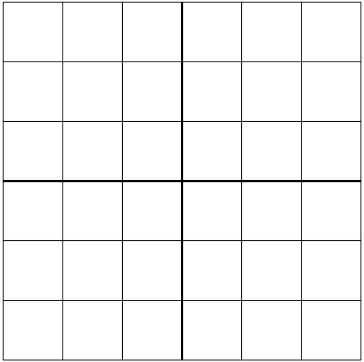
- Writing a vector in coordinate form requires choosing a **basis**.
- The "default" is the **standard basis**: $\hat{e}^{(1)}, \ldots, \hat{e}^{(d)}$.



Standard Basis

- When we write $\vec{x} = (x_1, \ldots, x_d)^T$, we mean that $\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)} + \ldots x_d \hat{e}^{(d)}$.

Example: $\vec{x} = (3, -2)^T$



Standard Basis Coordinates

- In coordinate form:

$$\hat{e}^{(i)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

where the 1 appears in the ith place.

Exercise

Let $\vec{x} = (3, 7, 2, -5)^T$. What is $\vec{x} \cdot \hat{e}^{(4)}$?

Recall: the Dot Product

- The **dot product** of $\vec{u}$ and $\vec{v}$ is defined as:

  $$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos \theta$$

  where θ is the angle between $\vec{u}$ and $\vec{v}$.

- $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u}$ and $\vec{v}$ are orthogonal

Dot Product (Coordinate Form)

- In terms of coordinate vectors:

$$
\begin{aligned}
\vec{u} \cdot \vec{v} &= \vec{u}^T \vec{v} \\
&= \begin{pmatrix} u_1 & u_2 & \cdots & u_d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_d \end{pmatrix} \\
&=
\end{aligned}
$$

- This definition assumes the standard basis.

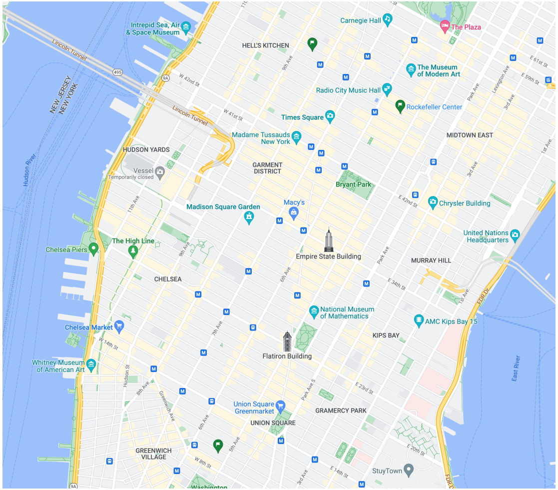
Example

$$\begin{pmatrix} 3 \\ 7 \\ 2 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$$

Other Bases

- The standard basis is not the **only** basis.
- Sometimes more convenient to use another.

Example



Intrepid Sea, Air & Space Museum
Carnegie Hall
The Plaza
HELL'S KITCHEN
Lincoln Tunnel
NEW JERSEY
NEW YORK
The Museum of Modern Art
Radio City Music Hall
Rockefeller Center
Times Square
Madame Tussauds New York
MIDTOWN EAST
HUDSON YARDS
GARMENT DISTRICT
Vessel
Temporarily closed
Hudson River
Bryant Park
Chrysler Building
Madison Square Garden
Macy's
United Nations Headquarters
The High Line
Chelsea Piers
Empire State Building
MURRAY HILL
CHELSEA
National Museum of Mathematics
AMC Kips Bay 15
KIPS BAY
Chelsea Market
Flatiron Building
Whitney Museum of American Art
East River
Union Square Greenmarket
GRAMERCY PARK
UNION SQUARE
GREENWICH VILLAGE
StuyTown

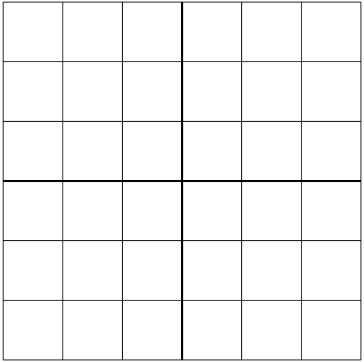
Orthonormal Bases

- **Orthonormal bases** are particularly nice.

- A set of vectors $\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}$ forms an **orthonomal basis** $\mathcal{U}$ for R^d if:
  - They are mutually orthogonal: $\hat{u}^{(i)} \cdot \hat{u}^{(j)} = 0$.
  - They are all unit vectors: $\|\hat{u}^{(i)}\| = 1$.

Example

$$\hat{u}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \hat{u}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$$



Coordinate Vectors

- A vector's coordinates depend on the basis used.
- If we are using the basis $\mathcal{U} = \{\hat{u}^{(1)}, \hat{u}^{(2)}\}$, then $\vec{x} = (x_1, x_2)^T$ means $\vec{x} = x_1 \hat{u}^{(1)} + x_2 \hat{u}^{(2)}$.
- We will write $[\vec{x}]_{\mathcal{U}} = (x_1, \ldots, x_d)^T$ to denote that the coordinates are with respect to the basis $\mathcal{U}$.

Exercise

Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$. Suppose $[\vec{x}]_{\mathcal{U}} = (3, -4)^T$. What is $\vec{x} \cdot \hat{u}^{(1)}$?

Exercise

Consider $\vec{x} = (2, 2)^T$ and let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$. What is $[\vec{x}]_{\mathcal{U}}$?

- A) $(0, 2\sqrt{2})^T$
- B) $(2, 2)^T$
- C) $(2\sqrt{2}, 0)^T$
- D) $(\sqrt{2}, \sqrt{2})^T$

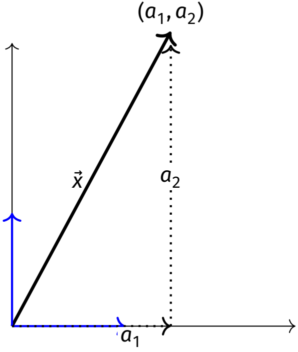
Change of Basis

- How do we compute the coordinates of a vector in a new orthonormal basis, $\mathcal{U}$?

- Some trigonometry is involved.

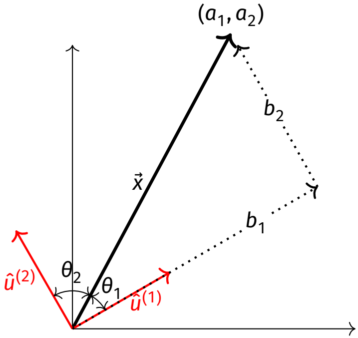
- **Key Fact**: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

Change of Basis



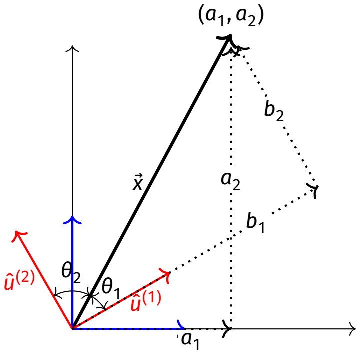
- Suppose we know $\vec{x} = (a_1, a_2)^T$ w.r.t. standard basis.
- Then $\vec{x} = a_1 \hat{e}^{(1)} + a_2 \hat{e}^{(2)}$

Change of Basis



- Want to write:
  $\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)}$
- Need to find b_1 and b_2.

Change of Basis



- **Exercise:** Solve for b_1, writing the answer as a dot product.
- Hint: $\cos\theta$ = adjacent/hypotenuse

Change of Basis

- Let $\mathcal{U} = \{\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}\}$ be an orthonormal basis.
- The coordinates of $\vec{x}$ w.r.t. $\mathcal{U}$ are:

$$[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} \vec{x} \cdot \hat{u}^{(1)} \\ \vec{x} \cdot \hat{u}^{(2)} \\ \vdots \\ \vec{x} \cdot \hat{u}^{(d)} \end{pmatrix}$$

Change of Basis

- Equivalently, to express $\vec{x}$ in basis $\mathcal{U}$:

$$\vec{x} = (\vec{x} \cdot \hat{u}^{(1)})\hat{u}^{(1)} + (\vec{x} \cdot \hat{u}^{(2)})\hat{u}^{(2)} + \ldots + (\vec{x} \cdot \hat{u}^{(d)})\hat{u}^{(d)}$$

Exercise

Suppose $\vec{x} = (2,1)^T$ and let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1,1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1,1)^T$. What is $[\vec{x}]_{\mathcal{U}}$?

- A) $\left(\frac{3\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)^T$
- B) $\left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)^T$
- C) $(2,1)^T$
- D) $\left(\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$

Exercise

Let $\vec{x} = (-1, 2)^T$ and suppose:

$$\hat{u}^{(1)} \cdot \hat{e}^{(1)} = \frac{3}{5} \qquad \hat{u}^{(2)} \cdot \hat{e}^{(1)} = -\frac{4}{5}$$

$$\hat{u}^{(1)} \cdot \hat{e}^{(2)} = \frac{4}{5} \qquad \hat{u}^{(2)} \cdot \hat{e}^{(2)} = \frac{3}{5}$$

What is $[\vec{x}]_{\mathcal{U}}$?

- A) $(1, 2)^T$
- B) $(2, 1)^T$
- C) $(-1, 2)^T$
- D) $(5, 10)^T$

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Lecture 02 | Part 4

Functions of a Vector

Functions of a Vector

- In ML, we often work with functions of a vector: $f : \mathbb{R}^d \to \mathbb{R}^{d'}$.

- Example: a prediction function, $H(\vec{x})$.

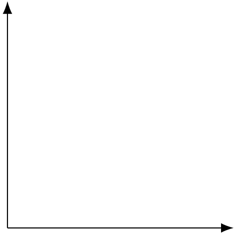
- Functions of a vector can return:
  - a number: $f : \mathbb{R}^d \to \mathbb{R}^1$
  - a vector $\vec{f} : \mathbb{R}^d \to \mathbb{R}^{d'}$
  - something else?

Transformations

- A **transformation** $\vec{f}$ is a function that takes in a vector, and returns a vector *of the same dimensionality*.
- That is, $\vec{f} : \mathbb{R}^d \to \mathbb{R}^d$.

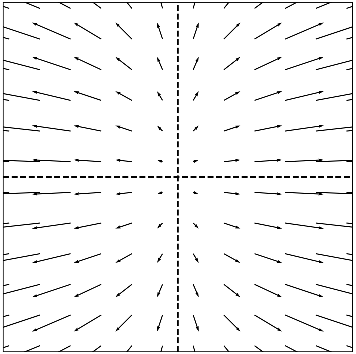
Visualizing Transformations

- A transformation is a **vector field**.
  - Assigns a vector to each point in space.
  - Example: $\vec{f}(\vec{x}) = (3x_1, x_2)^T$



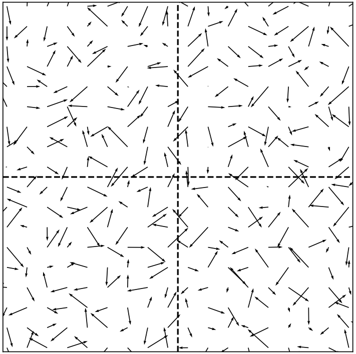
Example

- $\vec{f}(\vec{x}) = (3x_1, x_2)^T$



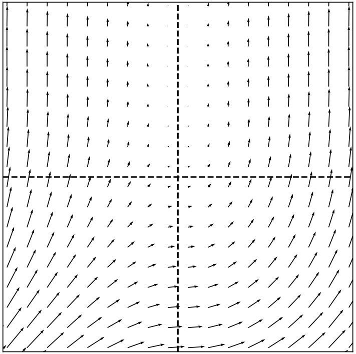
Arbitrary Transformations

- Arbitrary transformations can be quite complex.



Arbitrary Transformations

- Arbitrary transformations can be quite complex.



Linear Transformations

- Luckily, we often[2] work with simpler, **linear transformations**.

- A transformation f is **linear** if:

$$\vec{f}(\alpha\vec{x} + \beta\vec{y}) = \alpha\vec{f}(\vec{x}) + \beta\vec{f}(\vec{y})$$

[2]Sometimes just to make the math tractable!

Checking Linearity

- To check if a transformation is linear, use the definition.

- **Example:** $\vec{f}(\vec{x}) = (x_2, -x_1)^T$

Exercise

Let $\vec{f}(\vec{x}) = (x_1 + 3, x_2)$. True or False: $\vec{f}$ is a linear transformation.

Solution

- **False.** $\vec{f}$ is not a linear transformation.
- To see this, note that any linear transformation must satisfy $\vec{f}(\vec{0}) = \vec{0}$.
- However, $\vec{f}(\vec{0}) = (0 + 3, 0)^T = (3, 0)^T \neq \vec{0}$.
- Therefore, $\vec{f}$ is not linear.

Implications of Linearity

- Suppose $\vec{f}$ is a linear transformation. Then:

$$\begin{aligned}\vec{f}(\vec{x}) &= \vec{f}(x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)}) \\ &= x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)})\end{aligned}$$

- I.e., $\vec{f}$ is **totally determined** by what it does to the basis vectors.

The Complexity of Arbitrary Transformations

- Suppose f is an **arbitrary** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$?

The **Simplicity** of Linear Transformations

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$?

Exercise

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (3, -4)^T$.
- What is $\vec{f}(\vec{x})$?

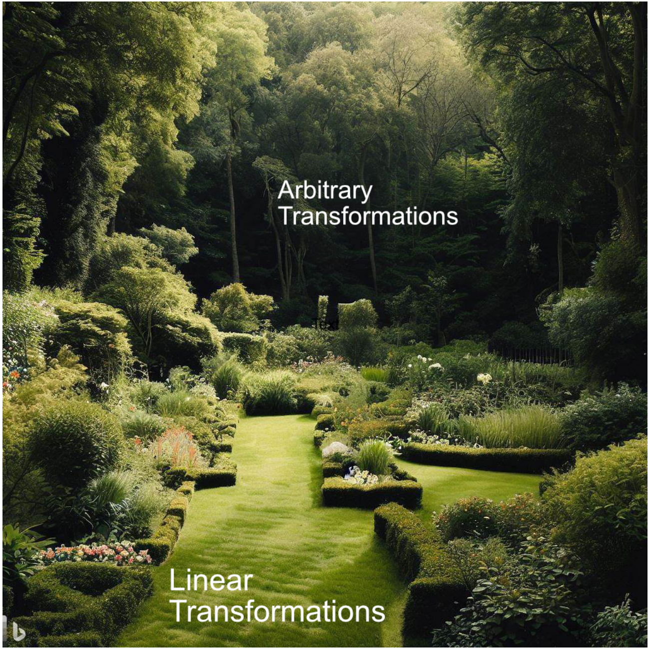
- A) $(3, 18)^T$
- B) $(6, 3)^T$
- C) $(-6, 3)^T$
- D) $(18, 3)^T$

Key Fact

- Linear functions are determined **entirely** by what they do on the basis vectors.
- I.e., to tell you what f does, I only need to tell you $\vec{f}(\hat{e}^{(1)})$ and $\vec{f}(\hat{e}^{(2)})$.
- This makes the math easy!

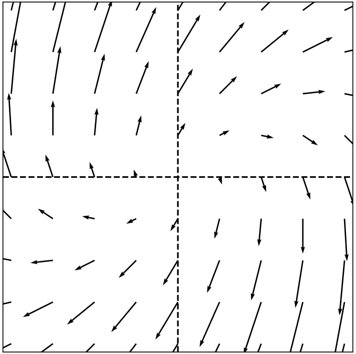
Linear Algebra

- This is the key idea behind **linear** algebra.
- Linear algebra studies the properties of **linear** transformations.
- Non-linear transformations are **so complicated** that we can say relatively little about them.


Arbitrary Transformations
Linear Transformations

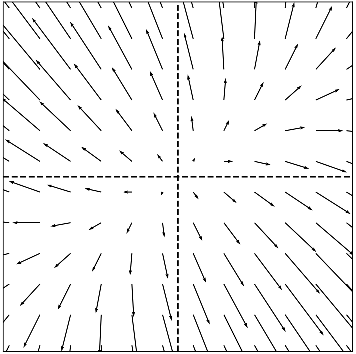
Example Linear Transformation

- $\vec{f}(\vec{x}) = (x_1 + 3x_2, -3x_1 + 5x_2)^T$



Another Example Linear Transformation

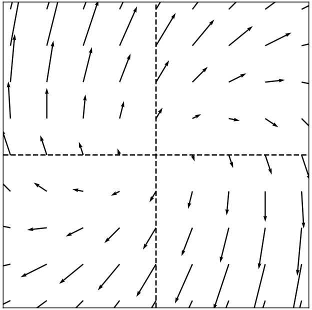
- $\vec{f}(\vec{x}) = (2x_1 - x_2, -x_1 + 3x_2)^T$

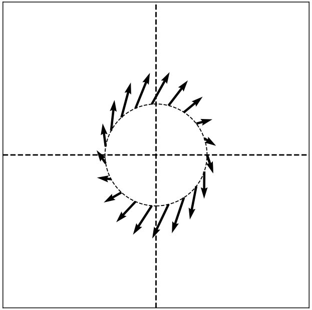


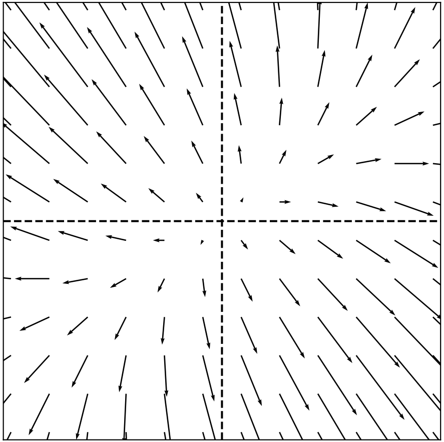
Note

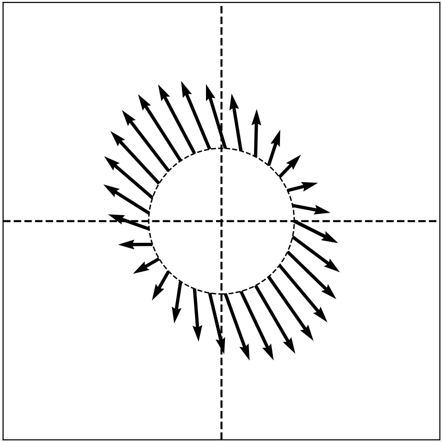
- Because of linearity, along any given direction $\vec{f}$ changes only in scale.

$$\vec{f}(\lambda\hat{x}) = \lambda\vec{f}(\hat{x})$$









Linear Transformations and Bases

- We have been writing transformations in coordinate form. For example:

$$\begin{aligned}\vec{f}(\vec{x}) &= (x_1 + x_2, x_1 - x_2)^T \\ &= (x_1 + x_2)\hat{e}^{(1)} + (x_1 - x_2)\hat{e}^{(2)}\end{aligned}$$

- If we use a different basis, the formula for $\vec{f}$ **changes**:

$$\begin{aligned}[\vec{f}(\vec{x})]_{\mathcal{U}} &= (?, ?)^T \\ &= [?]\hat{u}^{(1)} + [?]\hat{u}^{(2)}\end{aligned}$$

Linear Transformations and Bases

- We know that if $\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)}$, then:

$$\vec{f}(\vec{x}) = (x_1 + x_2)\hat{e}^{(1)} + (x_1 - x_2)\hat{e}^{(2)}$$

- Now: if $\vec{x} = z_1 \hat{u}^{(1)} + z_2 \hat{u}^{(2)}$, what is:

$$\vec{f}(\vec{x}) = ?\hat{u}^{(1)} + ?\hat{u}^{(2)}$$

Key Fact

- If we use linearity:

$$\begin{aligned} f(\vec{x}) &= f(z_1 \hat{u}^{(1)} + z_2 \hat{u}^{(2)}) \\ &= z_1 f(\hat{u}^{(1)}) + z_2 f(\hat{u}^{(2)}) \end{aligned}$$

- **Strategy**: to write $\vec{f}$ in the $\mathcal{U}$ basis, we just need to know what $\vec{f}$ does to $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$.

Example

- Let:
  - $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$
  - $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$.
- Then:

$$\vec{f}(\hat{u}^{(1)}) = \vec{f}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T = \left(\sqrt{2}, 0\right)^T = \sqrt{2}\hat{e}^{(1)}$$

$$\vec{f}(\hat{u}^{(2)}) = \vec{f}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T = \left(0, -\sqrt{2}\right)^T = -\sqrt{2}\hat{e}^{(2)}$$

- **But** we want $\vec{f}(\hat{u}^{(1)})$ and $\vec{f}(\hat{u}^{(2)})$ in terms of $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$.

Example (Cont.)

- We have: $f(\hat{u}^{(1)}) = \sqrt{2}\hat{e}^{(1)}$ and $f(\hat{u}^{(2)}) = -\sqrt{2}\hat{e}^{(2)}$.
- To write $\vec{f}(\hat{u}^{(1)})$ in terms of $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$, compute:

$$\begin{aligned} f(\hat{u}^{(1)}) &= (f(\hat{u}^{(1)}) \cdot \hat{u}^{(1)})\hat{u}^{(1)} + (f(\hat{u}^{(1)}) \cdot \hat{u}^{(2)})\hat{u}^{(2)} \\ &= \\ &= \end{aligned}$$

Example (Cont.)

- We have: $f(\hat{u}^{(1)}) = \sqrt{2}\hat{e}^{(1)}$ and $f(\hat{u}^{(2)}) = -\sqrt{2}\hat{e}^{(2)}$.
- To write $\vec{f}(\hat{u}^{(1)})$ in terms of $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$, compute:

$$\begin{aligned} f(\hat{u}^{(1)}) &= (f(\hat{u}^{(1)}) \cdot \hat{u}^{(1)})\hat{u}^{(1)} + (f(\hat{u}^{(1)}) \cdot \hat{u}^{(2)})\hat{u}^{(2)} \\ &= \left((\sqrt{2}, 0) \cdot \frac{1}{\sqrt{2}}(1, 1)\right)\hat{u}^{(1)} + \left((\sqrt{2}, 0) \cdot \frac{1}{\sqrt{2}}(-1, 1)\right)\hat{u}^{(2)} \\ &= \end{aligned}$$

Example (Cont.)

- We have: $f(\hat{u}^{(1)}) = \sqrt{2}\hat{e}^{(1)}$ and $f(\hat{u}^{(2)}) = -\sqrt{2}\hat{e}^{(2)}$.

- To write $\vec{f}(\hat{u}^{(1)})$ in terms of $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$, compute:

$$
\begin{aligned}
f(\hat{u}^{(1)}) &= (f(\hat{u}^{(1)}) \cdot \hat{u}^{(1)})\hat{u}^{(1)} + (f(\hat{u}^{(1)}) \cdot \hat{u}^{(2)})\hat{u}^{(2)} \\
&= \left((\sqrt{2}, 0) \cdot \frac{1}{\sqrt{2}}(1, 1)\right)\hat{u}^{(1)} + \left((\sqrt{2}, 0) \cdot \frac{1}{\sqrt{2}}(-1, 1)\right)\hat{u}^{(2)} \\
&= (1)\hat{u}^{(1)} + (-1)\hat{u}^{(2)} = \hat{u}^{(1)} - \hat{u}^{(2)}
\end{aligned}
$$

Example (Cont.)

- Similarly, for $\vec{f}(\hat{u}^{(2)})$:

$$\begin{aligned} f(\hat{u}^{(2)}) &= (f(\hat{u}^{(2)}) \cdot \hat{u}^{(1)})\hat{u}^{(1)} + (f(\hat{u}^{(2)}) \cdot \hat{u}^{(2)})\hat{u}^{(2)} \\ &= \left((0, -\sqrt{2}) \cdot \frac{1}{\sqrt{2}}(1, 1)\right)\hat{u}^{(1)} + \left((0, -\sqrt{2}) \cdot \frac{1}{\sqrt{2}}(-1, 1)\right)\hat{u}^{(2)} \\ &= (-1)\hat{u}^{(1)} + (-1)\hat{u}^{(2)} = -\hat{u}^{(1)} - \hat{u}^{(2)} \end{aligned}$$

Solution

- Putting it all together:

$$\begin{aligned} f(\vec{x}) &= f(z_1 \hat{u}^{(1)} + z_2 \hat{u}^{(2)}) \\ &= z_1 f(\hat{u}^{(1)}) + z_2 f(\hat{u}^{(2)}) \\ &= z_1(\hat{u}^{(1)} - \hat{u}^{(2)}) + z_2(-\hat{u}^{(1)} - \hat{u}^{(2)}) \\ &= (z_1 - z_2)\hat{u}^{(1)} + (-z_1 - z_2)\hat{u}^{(2)} \end{aligned}$$

- Or, in coordinate form:

$$[f(\vec{x})]_{\mathcal{U}} = (z_1 - z_2, -z_1 - z_2)^T$$

DSC 140B

Representation Learning

Lecture 02 | Part 5

Matrices

Matrices?

- I thought this week was supposed to be about linear algebra... Where are the matrices?

Matrices?

- I thought this week was supposed to be about linear algebra... Where are the matrices?
- What is a matrix, anyways?

What is a matrix?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Recall: Linear Transformations

- A **transformation** $\vec{f}(\vec{x})$ is a function which takes a vector as input and returns a vector of the same dimensionality.

- A transformation $\vec{f}$ is **linear** if

$$\vec{f}(\alpha\vec{u} + \beta\vec{v}) = \alpha\vec{f}(\vec{u}) + \beta\vec{f}(\vec{v})$$

Recall: Linear Transformations

- **Key** consequence of **linearity**: to compute $\vec{f}(\vec{x})$, only need to know what $\vec{f}$ does to basis vectors.

- Example:

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)}$$

$$\vec{f}(\vec{x}) =$$

Matrices

- **Idea**: Since $\vec{f}$ is defined by what it does to basis, place $\vec{f}(\hat{e}^{(1)})$, $\vec{f}(\hat{e}^{(2)})$, ... into a table as columns
- This is the **matrix** representing[3] $\vec{f}$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

---

[3] with respect to the standard basis $\hat{e}^{(1)}, \hat{e}^{(2)}$

Example

Write the matrix representing $\vec{f}$ with respect to the standard basis, given:

$$\vec{f}(\hat{e}^{(1)}) = (1, 4, 7)^T$$
$$\vec{f}(\hat{e}^{(2)}) = (2, 5, 8)^T$$
$$\vec{f}(\hat{e}^{(3)}) = (3, 6, 9)^T$$

Exercise

Suppose $\vec{f}$ has the matrix below:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Let $\vec{x} = (-2, 1, 3)^T$. What is $\vec{f}(\vec{x})$?

- A) $(3, 12, 21)^T$
- B) $(-2, 1, 3)^T$
- C) $(6, 15, 24)^T$
- D) $(9, 15, 21)^T$

Main Idea

A square $(n \times n)$ matrix can be interpreted as a compact representation of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$.

What is matrix multiplication?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

A low-level definition

$$(A\vec{x})_i = \sum_{j=1}^{n} A_{ij} x_j$$

A low-level interpretation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

In general...

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{a}^{(1)} & \vec{a}^{(2)} & \vec{a}^{(3)} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \vec{a}^{(1)} + x_2 \vec{a}^{(2)} + x_3 \vec{a}^{(3)}$$

Matrix Multiplication

$$\vec{x} = x_1\hat{e}^{(1)} + x_2\hat{e}^{(2)} + x_3\hat{e}^{(3)} = (x_1, x_2, x_3)^T$$

$$\vec{f}(\vec{x}) = x_1\vec{f}(\hat{e}^{(1)}) + x_2\vec{f}(\hat{e}^{(2)}) + x_3\vec{f}(\hat{e}^{(3)})$$

$$A = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x_1\vec{f}(\hat{e}^{(1)}) + x_2\vec{f}(\hat{e}^{(2)}) + x_3\vec{f}(\hat{e}^{(3)})$$

Matrix Multiplication

- Matrix A represents a linear transformation $\vec{f}$
  - With respect to the standard basis
  - If we use a different basis, the matrix changes!

- Matrix multiplication $A\vec{x}$ **evaluates** $\vec{f}(\vec{x})$

What are they, *really*?

- Matrices are sometimes just tables of numbers.
- But they often have a deeper meaning.

Main Idea

A square $(n \times n)$ matrix can be interpreted as a compact representation of a linear transformation $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$.

What's more, if A represents $\vec{f}$, then $A\vec{x} = \vec{f}(\vec{x})$; that is, multiplying by A is the same as evaluating $\vec{f}$.

Example

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)}$$

$$\vec{f}(\vec{x}) =$$

$$A =$$

$$A\vec{x} =$$

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.
- Linear algebra = simple!

Matrices in Other Bases

- The matrix of a linear transformation wrt the **standard basis**:

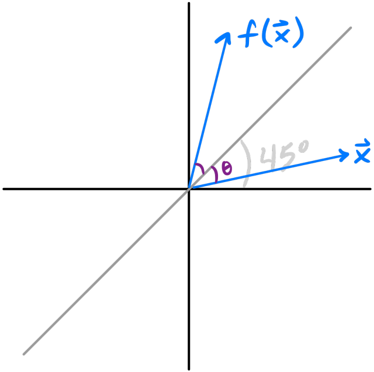
$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \cdots & \vec{f}(\hat{e}^{(d)}) \\ \downarrow & \downarrow & \downarrow & \end{pmatrix}$$

- With respect to basis $\mathcal{U}$:

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} & [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} & \cdots & [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow & \end{pmatrix}$$

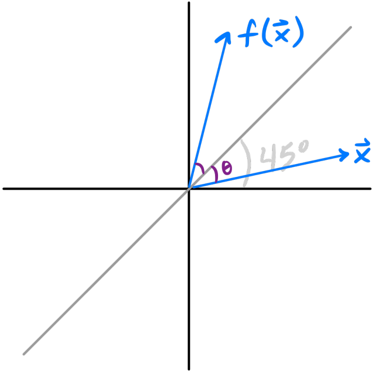
Example

- Consider the transformation $\vec{f}$ which “mirrors” a vector over the line of 45°.



- What is its matrix in the standard basis?

Example



- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$
- Let $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$
- What is $[\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}}$?
- $[\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}}$?
- What is the matrix?