DSC 40B - Sample Midterm 01

Note: This sample midterm is intended to give you an idea of the format of the exam, but it's not intended to be a comprehensive review of the material. Also, note that this sample exam is from a previous iteration of the course, and topics can change slightly from quarter to quarter depending on the instructor and how much was covered in lecture – you should expect Midterm 01 to cover the content from *this quarter*'s Lecture 01 through 08.

You should focus your studying on the practice problems found at `https://dsc140b.com/practice` that are labeled with `midterm-01`, as those will be the most representative of the types of questions you can expect to see on the exam.

Problem 1.
Let $\mathcal{U} = \{\vec{u}^{(1)}, \vec{u}^{(2)}, \vec{u}^{(3)}\}$ be an orthonormal basis, where:

$$\vec{u}^{(1)} = \frac{1}{\sqrt{2}}(-1, 1, 0)^T,$$
$$\vec{u}^{(2)} = \frac{1}{\sqrt{3}}(1, 1, 1)^T,$$
$$\vec{u}^{(3)} = \frac{1}{\sqrt{6}}(1, 1, -2)^T.$$

Let $\vec{x} = (1, 2, 3)^T$ be the coordinates of $\vec{x}$ with respect to the standard basis. What are the coordinates of $\vec{x}$ with respect to $\mathcal{U}$? That is, what is $[\vec{x}]_{\mathcal{U}}$?

$[\vec{x}]_{\mathcal{U}} = (\square , \square , \square)^T$

Problem 2.
Let $\mathcal{U} = \{\vec{u}^{(1)}, \vec{u}^{(2)}, \vec{u}^{(3)}\}$ be a basis for $\mathbb{R}^3$, where:

$$\vec{u}^{(1)} = (1, 0, 0)^T$$
$$\vec{u}^{(2)} = (0, 1, 1)^T$$
$$\vec{u}^{(3)} = (1, -1, 1)^T$$

Note that $\mathcal{U}$ is *not* an orthonormal basis.

Suppose that the coordinates of a vector $\vec{x}$ with respect to the basis $\mathcal{U}$ are given by $[\vec{x}]_{\mathcal{U}} = (1, 2, 3)^T$. What are the coordinates of $\vec{x}$ with respect to the standard basis?

$\vec{x} = (\square , \square , \square)^T$

Problem 3.
True or false; the transformation $\vec{f} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(\vec{x}) = (1 + x_1, 2x_2, x_1 - x_2)^T$, where $\vec{x} = (x_1, x_2, x_3)^T$ are the coordinates of $\vec{x}$ expressed in the standard basis, is a linear transformation.

◯ True

◯ False

Problem 4.

Let $\vec{f}(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ and $\vec{g}(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformations. Define $\vec{h}(\vec{x}) = \vec{f}(\vec{g}(\vec{x}))$.

True or false: $\vec{h}$ is a linear transformation.

◯ True

◯ False

Problem 5.

Let $\vec{f}(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose that the result of applying $\vec{f}$ to the standard basis vectors is:

$$\begin{aligned}\vec{f}(\hat{e}^{(1)}) &= 4\hat{e}^{(2)} - 3\hat{e}^{(3)}\\ \vec{f}(\hat{e}^{(2)}) &= 2\hat{e}^{(1)} - 2\hat{e}^{(2)} - \hat{e}^{(3)}\\ \vec{f}(\hat{e}^{(3)}) &= \hat{e}^{(1)} - 4\hat{e}^{(2)} + 2\hat{e}^{(3)}\end{aligned}$$

Let $\vec{x} = (2, -1, 3)^T$. Find $\vec{f}(\vec{x})$ as a coordinate vector with respect to the standard basis.

$\vec{f}(\vec{x}) = \Big(\boxed{}\ ,\ \boxed{}\ ,\ \boxed{} \Big)^T$

Problem 6.

Let $\vec{f}(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose that the result of applying $\vec{f}$ to the standard basis vectors is:

$$\begin{aligned}\vec{f}(\hat{e}^{(1)}) &= 4\hat{e}^{(2)} - 3\hat{e}^{(3)}\\ \vec{f}(\hat{e}^{(2)}) &= 2\hat{e}^{(1)} - 2\hat{e}^{(2)} - \hat{e}^{(3)}\\ \vec{f}(\hat{e}^{(3)}) &= \hat{e}^{(1)} - 4\hat{e}^{(2)} + 2\hat{e}^{(3)}\end{aligned}$$

Let A be the matrix of $\vec{f}$ with respect to the standard basis. What is the entry in the second row and third column of A?

$A_{23} = \boxed{}$

Problem 7.

Let $\vec{u}$ be a unit vector in $\mathbb{R}^d$. Consider the matrix U defined as $U = \vec{u}\vec{u}^T$. True or False: $\vec{u}$ is an eigenvector of U.

◯ True

◯ False

Problem 8. (2 points)

Let A be the matrix

$$A = \begin{pmatrix} 7 & 4 \\ 4 & 13 \end{pmatrix}.$$

It can be checked that the vector $\vec{u}^{(1)} = (1, 2)^T$ is an eigenvector of A.

a) What is the eigenvalue associated with $\vec{u}^{(1)}$?

[ ]

b) Find another eigenvector $\vec{u}^{(2)}$ of A. Your eigenvector should have an eigenvalue that is different from $\vec{u}^{(1)}$'s eigenvalue. It does not need to be normalized.

$\vec{u}^{(2)} = \big($ [ ] , [ ] $\big)^T$

Problem 9.

Let A be a symmetric matrix. Suppose the top eigenvalue of A is λ. Let $B = 3A$; that is, it is the matrix obtained by multiplying each entry of A by 3.

True or false: the top eigenvalue of B must be 3λ.

- ○ True
- ○ False

Problem 10.

Let A be a symmetric $d \times d$ matrix, and let U be a matrix whose rows are normalized eigenvectors of A. You may assume that all of the rows of U are mutually orthonormal.

True or False: U must be a diagonal matrix.

- ○ True
- ○ False

Problem 11.

Suppose $C = \begin{pmatrix} 5 & -3 \\ -3 & 6 \end{pmatrix}$ is the empirical covariance matrix for a data set of points. What is the variance in the direction given by the unit vector $\vec{u} = \frac{1}{\sqrt{2}}(-1, 1)^T$?

[ ]

Problem 12.

Let C be the sample covariance matrix of a data set, and suppose $\vec{u}^{(1)}, \vec{u}^{(2)}, \vec{u}^{(3)}$ are eigenvectors of C with eigenvalues $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 2$, respectively.

Suppose $\vec{x} = \frac{1}{\sqrt{3}}\vec{u}^{(1)} + \frac{2}{\sqrt{6}}\vec{u}^{(2)} + \frac{1}{\sqrt{2}}\vec{u}^{(3)}$. What is the variance in the direction of $\vec{x}$?

[ ]

Problem 13. (2 points)

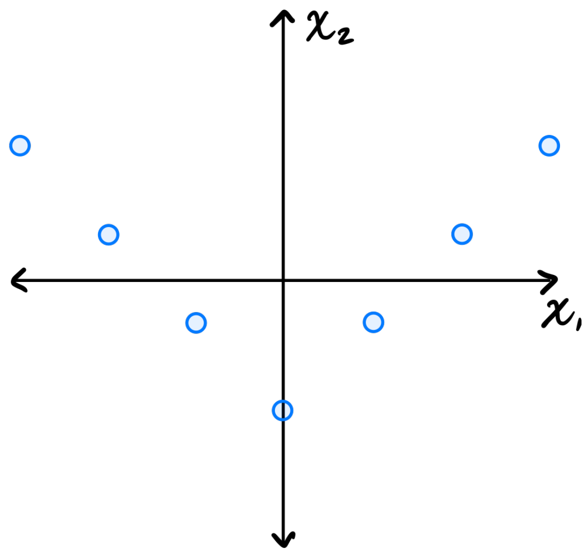
Consider the dataset of four points in $\mathbb{R}^2$ shown below:

$$\vec{x}^{(1)} = (12, 2)^T$$
$$\vec{x}^{(2)} = (8, 6)^T$$
$$\vec{x}^{(3)} = (6, 5)^T$$
$$\vec{x}^{(4)} = (14, 3)^T$$

Calculate the sample covariance matrix, and show your work.

Problem 14.

Consider the data set in the image shown below:



You may assume that this data is already centered and that the symmetry over the x_2 axis is exact. Which one of the following is true about the $(1, 2)$ entry of the data's sample covariance matrix?

- ◯ It is positive (>0).
- ◯ It is zero.
- ◯ It is negative (<0).

Problem 15.

Let $\mathcal{X} = \{\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}\}$ be a data set of n points in $\mathbb{R}^2$. Consider the linear transformation $\vec{f}(\vec{x}) = (-x_1, x_2)^T$, which takes an input point and "flips" it over the x_2 axis. Let $\mathcal{X}' = \{\vec{f}(\vec{x}^{(1)}), \ldots, \vec{f}(\vec{x}^{(n)})\}$ be the set of points in $\mathbb{R}^2$ obtained by applying $\vec{f}$ to each point in $\mathcal{X}$.

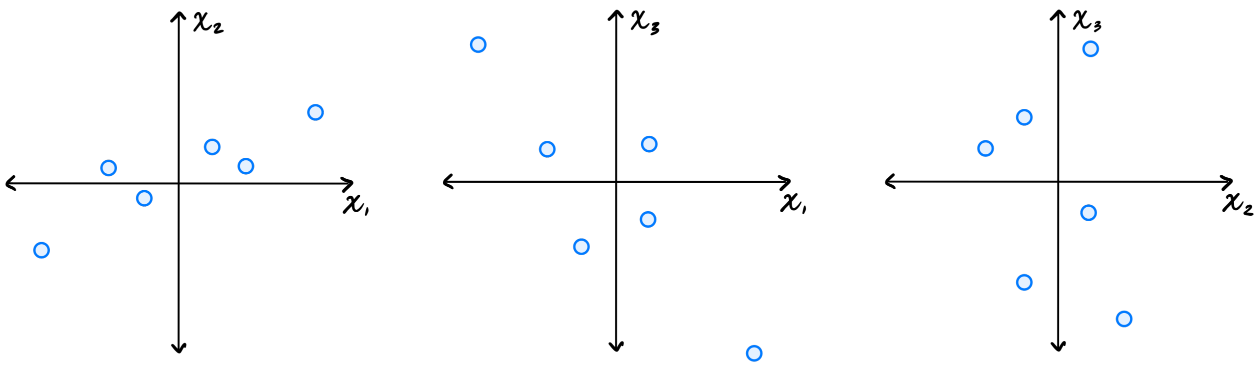
Suppose the sample covariance matrix of $\mathcal{X}$ is

$$C = \begin{pmatrix} 5 & -3 \\ -3 & 4 \end{pmatrix}.$$

What is the sample covariance matrix of $\mathcal{X}'$? You do not need to show your work.

Problem 16.

Let $\vec{x}^{(1)}, \ldots, \vec{x}^{(6)}$ be a data set of 6 points in $\mathbb{R}^3$. Shown below are scatter plots of each pair of coordinates (pay close attention to the axis labels):



Which one of the following could possibly be the data's sample covariance matrix?

- ◯ $\begin{pmatrix} 9 & 3 & -1 \\ 3 & 12 & 2 \\ -1 & 2 & 11 \end{pmatrix}$
- ◯ $\begin{pmatrix} 8 & -3 & 2 \\ -3 & 6 & 1 \\ 2 & 1 & 7 \end{pmatrix}$
- ◯ $\begin{pmatrix} 10 & 4 & -2 \\ 4 & 5 & 0 \\ -2 & 0 & 10 \end{pmatrix}$
- ◯ $\begin{pmatrix} 12 & 2 & 6 \\ 2 & 8 & -4 \\ 6 & -4 & 9 \end{pmatrix}$

Problem 17.

Suppose C is a 3×3 sample covariance matrix for a data set $\mathcal{X}$, and that the top two eigenvectors of C are:

$$\vec{u}^{(1)} = \frac{1}{\sqrt{2}}(-1, 1, 0)^T,$$
$$\vec{u}^{(2)} = \frac{1}{\sqrt{3}}(1, 1, 1)^T,$$

with eigenvalues $\lambda_1 = 10$ and $\lambda_2 = 4$, respectively.

Let $\vec{x} = (1, 2, 3)^T$ be the coordinates of $\vec{x}$ with respect to the standard basis. Let $\vec{z}$ be the result of applying PCA to reduce the dimensionality of $\vec{x}$ to 2. What is $\vec{z}$?

$\vec{z} = ($ ☐ , ☐ $)^T$

Problem 18.

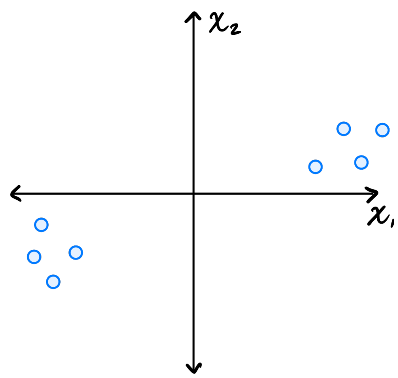
Let C be the sample covariance matrix of a data set $\mathcal{X}$ consisting of five points. Suppose that PCA is performed to reduce the dimensionality of $\mathcal{X}$ to one dimension. The results are:

$$\begin{aligned} z^{(1)} &= 4 \\ z^{(2)} &= 3 \\ z^{(3)} &= -2 \\ z^{(4)} &= 1 \\ z^{(5)} &= -6 \end{aligned}$$

What is the largest eigenvalue of C?

Problem 19.

Consider the data set shown below:



Which of the following could possibly be the top eigenvector of the data's sample covariance matrix?

- ◯ $(1, 0)^T$
- ◯ $(0, 1)^T$
- ◯ $(1, 1)^T$
- ◯ $(3, 1)^T$

Problem 20.

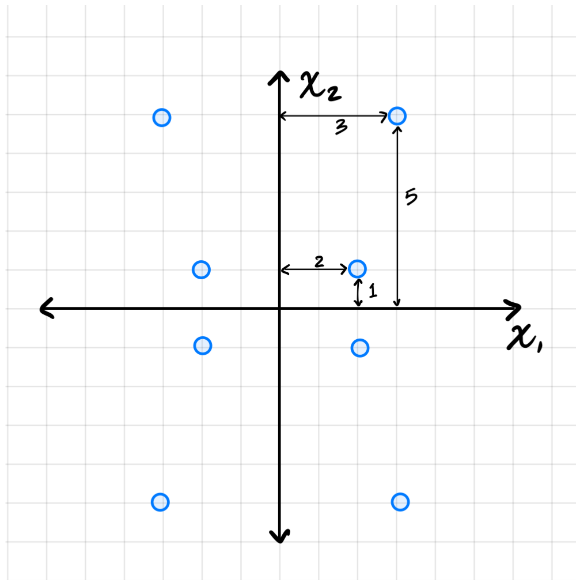
Let $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ be two points in a data set $\mathcal{X}$ of 100 points in d dimensions. Suppose PCA is performed, but dimensionality is not reduced; that is, each point $\vec{x}^{(i)}$ is transformed to the vector $\vec{z}^{(i)} = U\vec{x}^{(i)}$, where U is a matrix whose d rows are the (orthonormal) eigenvectors of the data covariance matrix.

True or False: $\|\vec{x}^{(1)} - \vec{x}^{(2)}\| = \|\vec{z}^{(1)} - \vec{z}^{(2)}\|$. That is, the distance between $\vec{z}^{(1)}$ and $\vec{z}^{(2)}$ in the new data set is necessarily the same as the distance between their corresponding points in the original data set.

- ◯ True
- ◯ False

Problem 21.

Suppose PCA is used to reduce the dimensionality of the data shown below from 2 dimensions to 1.



What will be the reconstruction error?

Problem 22.

Consider again the data set shown in the last problem. What is the smallest eigenvalue of the data's covariance matrix?

Problem 23.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two data sets containing 100 points each, and let $\mathcal{X}$ be the combination of the two data sets into a data set of 200 points.

Suppose $\vec{x} \in \mathcal{X}_1$ is a point in the first data set. Suppose PCA is performed on $\mathcal{X}_1$ by itself, reducing each point to one dimension, and that the new representation of $\vec{x}$ is z.

The point $\vec{x}$ is also in the combined data set, $\mathcal{X}$. Suppose PCA is performed on the combined data set, $\mathcal{X}$, reducing each point to one dimension, and that the new representation of $\vec{x}$ after this PCA is z'.

True or False: it is necessarily the case that $z = z'$.

- ◯ True
- ◯ False

Before turning in your exam, please check that your name is on every page.

(You may use this area for scratch work.)